Approach to coefficient Inequality for a new subclass of Starlike Function with extremals

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Abstract- The aim of the present paper is to investigate a certain subclass $S^*(A, B, p, \delta)$ of starlike function and obtain the sharp upper bound of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $|z| < 1$ belonging to this subclass of starlike function.

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1 INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

(1)

Which are analytic in the open unit disc $U = \{z : z \in \mathbb{C}; |z| < 1\}$ and let $S$ denote the class of functions in $A$ that are univalent in $U$.

In 1916, for the functions $f(z) \in S$, Bieber Bach [4, 5] proved the result $|a_2| \leq 2$. In 1923, for the same functions, Lowner [2] proved that $|a_3| \leq 3$. With these results $|a_2| \leq 2$ and $|a_3| \leq 3$, for the class $S$ it was very easy to draw out the relation between $a_3$ and $a_2^2$. With the help of Lowner’s method, Fekete and Szego [6] proved the following well known resul

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0 \\ 1 + \exp\left(-\frac{2\mu}{1 - \mu}\right) & \text{if } 0 \leq \mu \leq 1 \\ 4\mu - 3 & \text{if } \mu \geq 1 \end{cases} \]

This inequality is very much helpful in determining estimates of higher coefficients for some subclasses $S$ (See Chhichra [1], Babalola [3]).

Now we define some subclasses of $S$. Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in A$$

and satisfy the condition

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U$$

is univalent starlike function and denoted by
\( S^* \) and a subclass

\[
S^*(A,B) = \left\{ f(z) \in A, \frac{zf'(z)}{f(z)} < \frac{1 + A z}{1 + B z} \right\}
\]

where

\[-1 \leq B < A \leq 1, \ z \in U\]

It is obvious that \( S^*(A,B) \) is a subclass of \( S^* \).

We introduce a new class as

\[
S^*(p) = \left\{ f(z) \in A, \frac{zf'(pz)}{pf(z)} < \frac{1 + z}{1 - z}, \ z \in U \right\}
\]

Symbol \( \prec \) stands for subordination.

**Analytic bounded functions:** Class of analytic bounded function is of the form

\[
w(z) = \sum_{n=1}^{\infty} c_{n} z^{n}, w(0) = 0, |w(z)| \leq 1.
\]

It is known that \( |c_{1}| \leq 1, |c_{2}| \leq 1 - |c_{1}|^{2} \).

**2 FEKETE-SZEGO PROBLEM**

Our main result is the following

**2.1 Theorem**

Let the bounded function \( w(z) = c_{1} z + c_{2} z^{2} + \cdots \) and \( f(z) \in S^*(A, B, p, \delta) \), then

\[
\frac{zf'(pz)}{pf(z)} \prec \left( \frac{1 + Aw(z)}{1 + Bw(z)} \right)^{6} \quad \ldots (2)
\]

By expending the series (2)
\[
1 + (2p - 1)a_2 z + \left[(1 - 2p)a_2^2 + (3p^2 - 1)a_3\right]z^2 + \cdots = 1 + \frac{\delta(A - B)c_1}{2p - 1} + \left[\frac{\delta(A - B)c_2}{2p - 1} + \frac{\delta(B^2 - AB)c_1^2}{2p - 1}\right]z^2 + \cdots
\]

Comparing coefficients of (3)

\[
a_2 = \frac{\delta(A - B)c_1}{2p - 1}
\]

and

\[
a_3 = \frac{(A - B)\delta}{3p^2 - 1} c_2 + \frac{\delta c_1^2}{2p - 1}\left[\frac{4Bp(B - A)}{3p^2 - 1} + (\delta - 1)(2A(B + Ap) - B^2)\right] + (3 - \delta)A^2 - 2AB
\]

\[
\cdots(4)
\]

\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{3p^2 - 1}|c_1|^2 + \frac{4Bp(B - A)}{3p^2 - 1}(2p - 1) + (\delta - 1)(2A(B + Ap) - B^2) + (3 - \delta)A^2 - 2AB
\]

Case 1: when \[
\mu \leq \frac{(2p - 1)}{2(3p^2 - 1)(A - B)^2} + (\delta - 1)(2A(B + Ap) - B^2)
\]

Inequality (5) can be rewritten as

\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{3p^2 - 1} + \frac{4Bp(B - A)}{3p^2 - 1}(2p - 1) + (\delta - 1)(2A(B + Ap) - B^2) + (3 - \delta)A^2 - 2AB
\]
Sub case 1(a): When
\[
\mu \leq \frac{(2p - 1)}{(3p^2 - 1)(A - B)^2} \\
\left[ \delta(B - A)(4Bp + 2(2p - 1)) + \delta(\delta - 1)((2AB + 2A^2 p) - B^2) + (3 - \delta)A^2 - 2AB \right]
\]

Then equation (6) can be rewritten as
\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{(2p - 1)}
\]

Sub case 1(b): When
\[
\frac{(2p - 1)}{(3p^2 - 1)(A - B)^2} \\
\left[ \delta(B - A)(4Bp + 2(2p - 1)) + \delta(\delta - 1)((2AB + 2A^2 p) - B^2) + (3 - \delta)A^2 - 2AB \right]
\]

then the equation (6) becomes
\[
|a_3 - \mu a_2^2| \leq \frac{\delta(A - B)}{3p^2 - 1}
\]... (8)
\[
\frac{2(p-1)}{3p^2-1} \left[ \begin{array}{c}
4Bp \delta(B - A) \\
(\delta - 1)(2A(B + Ap) - B^2) \\
+ (3 - \delta)A^2 - 2AB \\
\end{array} \right]
\]

< \mu <

\[
\frac{2(p-1)}{3p^2-1} \left[ \begin{array}{c}
\delta(2B(A - B) - (2p-1)) \\
\delta(2AB + 2A^2 p) - B^2) \\
- (3 - \delta)A^2 + 2AB \\
\end{array} \right]
\]

Then the equation (9) becomes

\[
|a_3 - \mu a_2|^2 \leq \frac{\delta(A - B)}{3p^2 - 1} \quad \ldots (11)
\]

Combining the equations (7), (8), (10) and (11).

We get the Fekete Szego inequality for

\[ S^*(A, B, p, \delta) \]

as

\[
\begin{cases}
\frac{(A - B)}{2(p - 1)} \left[ \begin{array}{c}
\frac{U + V}{(A - B)^2} \\
\frac{2(3p^2 - 1)(A - B)}{(2p - 1)^2} \\
\end{array} \right] \\
\delta(A - B) \\
\end{cases}
\]

if \( \mu \leq \lambda_1; \)

\[
|a_2 - \mu a_3| \leq 
\begin{cases}
\frac{(A - B)}{2p - 1} \left\{ \frac{(A - B)}{2p - 1} + \frac{U + V}{(A - B)^2} \right\} \\
\delta(A - B) \\
\end{cases}
\]

if \( \lambda_1 < \mu < \lambda_2; \)

\[
\frac{(A - B)}{2p - 1} \left\{ \frac{(A - B)}{2p - 1} \right\} + \frac{U + V}{(A - B)^2} \\
\]

if \( \mu \geq \lambda_2. \)

The extremal function for first and third inequality is

\[ f_1(z) = z^{1 + \alpha z} \]

Where

\[ a = \frac{(A - B)^2(3p^2 - 1) - (U + V)(2p - 1)}{(A - B)(3p^2 - 1)(2p - 1)} \]

And extremal function for second inequality is

\[ f_2(z) = z^{1 + \delta(A - B)z^2} \]

Corollary 1: Putting \( A = 1, B = -1, \delta = 1 \) in the theorem 2.1 we get

\[
|a_2 - \mu a_3| \leq 
\begin{cases}
\frac{2(2p+1)}{3p^2 - 1} - \frac{4}{(2p-1)^2} \mu \quad \text{if} \quad \mu \leq \frac{2p-1}{3p^2 - 1}; \\
\frac{2}{3p^2 - 1} \quad \text{if} \quad \frac{2p-1}{3p^2 - 1} \leq \mu \leq \frac{2p(2p-1)}{3p^2 - 1} \\
\frac{4}{(2p-1)^2} - \frac{2(2p+1)}{3p^2 - 1} \quad \text{if} \quad \mu \geq \frac{2p(2p-1)}{3p^2 - 1} \\
\end{cases}
\]

Which is the result obtained by [9].

Corollary 2: Putting \( p = 1 \) and \( A = 1, B = -1, \delta = 1 \) in the theorem 2.1 we get

\[
|a_4 - \mu a_2|^2 \leq 
\begin{cases}
3 - 4\mu \quad \text{if} \quad \mu \leq \frac{1}{2}; \\
1 \quad \text{if} \quad \frac{1}{2} < \mu < 1; \\
4\mu - 3 \quad \text{if} \quad \mu \geq 1 \\
\end{cases}
\]

Which is the result obtained by [8].

REFERENCE