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Bipolar Pythagorean Fuzzy Sets and Their Application Based on Multi-Criteria Decision Making Problems

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Abstract - In this paper, we introduce a new concept of bipolar Pythagorean fuzzy set (BPFS) and some of its operations. Also, we propose score and accuracy functions to compare the bipolar Pythagorean fuzzy sets. Further we discuss bipolar Pythagorean fuzzy weighted average operator (A_w) and bipolar Pythagorean fuzzy weighted geometric operator (G_w) to aggregate the bipolar Pythagorean fuzzy information. Finally, we apply these operators to deal with multi-criteria decision making approach by using the bipolar Pythagorean fuzzy numbers.

Keywords: Pythagorean fuzzy set, bipolar Pythagorean fuzzy set, average operator, geometric operator, score, accuracy functions and multi-criteria decision making.

1. INTRODUCTION

Fuzzy sets were introduced by Zadeh [26] and he discussed only membership function.After the extensions of fuzzy set theory Atanassov [4] generalized this concept and introduced a new concept called intuitionistic fuzzy set (IFS). K.Atanassov [4,5,6,7] presented the idea of IFS. Gau and Buehrer [14] familiarized the concept of another set called vague set. Burilo and Bustin [9] developed a relation between the two famous sets called vague set and IFS. They also mathematically proved that these sets are equivalent. Yager [22] familiarized the model of Pythagorean fuzzy set. The most important and central research topic is aggregation operators. There are many scholars worked in this area and introduced several operators. IFS has its greatest use in practical multiple attribute decision making (MADM) problems, and the academic research have achieved great development [22,24,25].However, in the some practical problems, the sum of membership degree and non-membership degree to which an alternative satisfying an attribute provided by decision maker(DM) may be bigger than 1, but their square sum is less than or equal to1.

Bosc and Pivert [8] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints, since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes." Therefore, Lee [17,18] introduced the concept of bipolar fuzzy sets which is an generalization of the fuzzy sets. Recently, bipolar fuzzy models have been studied by many authors on algebraic structures such as; Chen et. al. [10] studied of m-polar fuzzy set. Then, they examined many results which are related to these concepts can be generalized to the case of m-polar fuzzy sets. They also proposed numerical examples to show how to apply m-polar fuzzy sets in real world problems.

In this paper, we introduce the concept of bipolar Pythagorean fuzzy sets which is an extension of the fuzzy sets,bipolar fuzzy sets,intuitionistic fuzzy sets and Pythagorean fuzzy sets. Also, we give some operators and operators on the bipolar Pythagorean fuzzy sets. We discuss the some operators based on BPFWA and BPFGA operators. We develop a bipolar Pythagorean fuzzy multiple criteria decision-making approach, in which the evaluation values of alternatives on the attributes take the form of bipolar Pythagorean fuzzy numbers to select the most desirable one(s) and give a numerical example.

2. PRELIMINARIES

Definition 2.1[26]

Let X be a non-empty set and I the unit interval [0,1]. A PF set S is an object having the form $P = \{\langle x, \mu_P(x), v_P(x)\rangle : x \in X\}$ where the functions $u_P: X \to [0,1]$ *and* $v_P: X \to [0,1]$ denote respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set P, and $0 \le (\mu_P(x))^2 + (v_P(x))^2 \le 1$ for each $x \in X$.

Definition 2.2[26]

Let X be a nonempty set and I the unit interval [0,1]. A PF sets P_1 and P_2 be in the form $P_1 = \{\langle x, \mu_{P_1}(x), v_{P_1}(x)\rangle : x \in X\}$and $P_2 = \{\langle x, \mu_{P_2}(x), v_{P_2}(x)\rangle : x \in X\}$.

Then

1) $P^C = \{\langle x, v_P(x), \mu_P(x)\rangle : x \in X\}$
2) $P_1 \cup P_2 = \left\{\langle x, \max\left(\mu_{P_1}(x), \mu_{P_2}(x)\right), \min(v_{P_1}(x), v_{P_2}(x))\rangle : x \in X\right\}$
3) $P_1 \cap P_2 = \left\{\langle x, \min\left(\mu_{P_1}(x), \mu_{P_2}(x)\right), \max(v_{P_1}(x), v_{P_2}(x))\rangle : x \in X\right\}$

Definition 2.3[26]

Let $P = (\mu_P, v_P)$, $P_1 = (\mu_{P_1}, v_{P_1})$, and $P_2 = (\mu_{P_2}, v_{P_2})$, be three PFNs and $\lambda > 0$, then their operations are defined as follows:

1) $P_1 \oplus P_2 = \left(\sqrt{\mu_{P_1}^2 + \mu_{P_2}^2 - \mu_{P_1}^2 \mu_{P_2}^2}, v_{P_1} v_{P_2}\right)$
2) $P_1 \otimes P_2 = \left(\mu_{P_1}\mu_{P_2}, \sqrt{v_{P_1}^2 + v_{P_2}^2 - v_{P_1}^2 v_{P_2}^2}\right)$
3) $\lambda P = \left(\sqrt{1-(1-\mu_P^2)^\lambda}, \mu_P^\lambda\right)$
4) $P^\lambda = \left(\mu_P^\lambda, \sqrt{1-(1-v_P^2)^\lambda}\right)$

Definition 2.4[26]

For any PFN the score function of P is defined as follows:

$$S(P) = \mu_P^2(x) - v_P^2(x)$$

where $S(P) \in [-1,1]$. For any two PFNs P_1, P_2 , if $S(P_1) < S(P_2)$, then $P_1 < P_2$. If $S(P_1) > S(P_2)$, then $P_1 > P_2$.If $S(P_1) = S(P_2)$, then $P_1 \sim P_2$.

Definition 2.5[26]

For any PFNs $P = (\mu_P, v_P)$, the accuracy function of A is defined as follows:

$$a(P) = \mu_P^2(x) + v_P^2(x)$$

where $a(P) \in [0,1]$.

3. BIPOLAR PYTHAGOREAN FUZZY SETS

Definition 3.1

Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $A = \{(x, (T_A^P, F_A^P), (T_A^N, F_A^N)) | x \in X\}$ where $T_A^P: X \to [0,1]$, $F_A^P: X \to [0,1]$, $T_A^N: X \to [-1,0]$, $F_A^N: X \to [-1,0]$ are the mappings such that

$$0 \le (T_A^P(x))^2 + (F_A^P(x))^2 \le 1$$

and $-1 \le -((T_A^N(x))^2 + (F_A^N(x))^2) \le 0$

and $T_A^P(x)$ denote the positive membership degree, $F_A^P(x)$ denote the positive non-membership degree, $T_A^N(x)$ denote the negative membership degree and $F_A^N(x)$ denote the negative non-membership degree. The degree of indeterminacy

$\pi_A^P(x) = \sqrt{1-(T_A^P(x))^2 - (F_A^P(x))^2}$ and

$\pi_A^N(x) = -\sqrt{1-(T_A^N(x))^2 - (F_A^N(x))^2}$.

Definition 3.2

For any BPFN the score function of A is defined as follows:

$$S(A) = \frac{1}{2}((T_A^P(x))^2 - (F_A^P(x))^2 + (T_A^N(x))^2 - (F_A^N(x))^2)$$

where $S(A) \in [-1,1]$. For any two BPFNs A,B , if $S(A) < S(B)$, then $A < B$. If $S(A) > S(B)$, then $A > B$.If $S(A) = S(B)$, then $A \sim B$.

Definition 3.3

For any BPFNs $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, the accuracy function of A is defined as follows:

$$a(A) = \frac{1}{2}((T_A^P(x))^2 + (F_A^P(x))^2 + (T_A^N(x))^2 + (F_A^N(x))^2)$$

where $a(A) \in [0,1]$.

Definition 3.4

Let $A = \{(x, (T_A^P, F_A^P), (T_A^N, F_A^N)) : x \in X\}$ and $B = \{(x, (T_B^P, F_B^P), (T_B^N, F_B^N)) : x \in X\}$be two BPFSs, then their operations are defined as follows:

(1)$A \cup B = \{(x, max(T_A^P, T_B^P), min(F_A^P, F_B^P), min(T_A^N, T_B^N), max(F_A^N, F_B^N)) : x \in X\}$

(2)$A \cap B = \{(x, min(T_A^P, T_B^P), max(F_A^P, F_B^P), max(T_A^N, T_B^N), min(F_A^N, F_B^N)) : x \in X\}$

(3)$A^C = \{(x, (F_A^P, T_A^P), (F_A^N, T_A^N)) : x \in X\}$

Definition 3.5

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs and $\lambda > 0$, then their operations are defined as follows:

(1) $A \oplus B = \begin{pmatrix} \left(\sqrt{(T_A^P)^2 + (T_B^P)^2 - (T_A^P)^2(T_B^P)^2}, F_A^P F_B^P\right), \\ \left(-T_A^N T_B^N, -\sqrt{(F_A^N)^2 + (F_B^N)^2 - (F_A^N)^2(F_B^N)^2}\right) \end{pmatrix}$

(2) $A \otimes B = \begin{pmatrix} \left(T_A^P T_B^P, \sqrt{(F_A^P)^2 + (F_B^P)^2 - (F_A^P)^2(F_B^P)^2}\right), \\ \left(-\sqrt{(T_A^N)^2 + (T_B^N)^2 - (T_A^N)^2(T_B^N)^2}, -F_A^N F_B^N\right) \end{pmatrix}$

(3) $\lambda A = \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^\lambda}, (F_A^P)^\lambda\right), \\ \left(-(-T_A^N)^\lambda, -\sqrt{1-(1-(F_A^N)^2)^\lambda}\right) \end{pmatrix}$

(4) $A^\lambda = \begin{pmatrix} \left((T_A^P)^\lambda, \sqrt{1-(1-(F_A^P)^2)^\lambda}\right), \\ \left(-\sqrt{1-(1-(T_A^N)^2)^\lambda}, -(-F_A^N)^\lambda\right) \end{pmatrix}$

Theorem 3.1

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then,

1) $A \oplus B = B \oplus A$;
2) $A \otimes B = B \otimes A$;
3) $\lambda(A \oplus B) = \lambda A \oplus \lambda B$;
4) $\lambda_1 A \oplus \lambda_2 A = (\lambda_1 + \lambda_2)A$;
5) $(A \otimes B)^\lambda = A^\lambda \otimes B^\lambda$;
6) $A^{\lambda_1} \otimes A^{\lambda_2} = A^{(\lambda_1+\lambda_2)}$;

Definition 3.6

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$ and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$ be two BPFNs, then

1) $A \ominus B = \begin{pmatrix} \left(\sqrt{\frac{(T_A^P)^2-(T_B^P)^2}{1-(T_B^P)^2}}, \frac{F_A^P}{F_B^P}\right), \\ \left(-\frac{T_A^N}{T_B^N}, -\sqrt{\frac{(F_A^N)^2-(F_B^N)^2}{1-(F_B^N)^2}}\right) \end{pmatrix}$

If $T_A^P \geq T_B^P$, $F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}$, $F_A^N \leq F_B^N$.

2) $A \oslash B = \begin{pmatrix} \left(\frac{T_A^P}{T_B^P}, \sqrt{\frac{(F_A^P)^2-(F_B^P)^2}{1-(F_B^P)^2}}\right), \\ \left(-\sqrt{\frac{(T_A^N)^2-(T_B^N)^2}{1-(T_B^N)^2}}, -\frac{F_A^N}{F_B^N}\right) \end{pmatrix}$

if $T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}$, $F_A^P \geq F_B^P$

$T_A^N \leq T_B^N$, $F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$.

Theorem 3.2

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then,

(1) $(A^C)^\lambda = (\lambda A)^C$;
(2) $\lambda(A^C) = (A^\lambda)^C$;
(3) $A \cup B = B \cup A$;
(4) $A \cap B = B \cap A$;
(5) $\lambda(A \cup B) = \lambda A \cup \lambda B$;
(6) $(A \cup B)^\lambda = A^\lambda \cup B^\lambda$;
(7) $(A \ominus B) = \lambda A \ominus \lambda B$;

$if\ \ T_A^P \geq T_B^P$, $F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}$, $F_A^N \leq F_B^N$

(8) $(A \oslash B)^\lambda = A^\lambda \oslash B^\lambda$;

$if\ \ T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}$, $F_A^P \geq F_B^P$,

$T_A^N \leq T_B^N$, $F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$.

(9) $\lambda_1 A \ominus \lambda_2 A = (\lambda_1 - \lambda_2)A$; $\ if\ \ \lambda_1 \geq \lambda_2$;
(10) $A^{\lambda_1} \oslash A^{\lambda_2} = A^{(\lambda_1-\lambda_2)}$.

Proof:

In the following, we shall prove (1),(3),(5),(7),(9) and (2),(4),(6),(8),(10) can be proved analogously.

(1) $(A^C)^\lambda = (F_A^P, T_A^P, F_A^N, T_A^N)^\lambda = \begin{pmatrix} \left((F_A^P)^\lambda, \sqrt{1-(1-(T_A^P)^2)^\lambda}\right), \\ \left(-\sqrt{1-(1-(F_A^N)^2)^\lambda}, -(-T_A^N)^\lambda\right) \end{pmatrix}$

$(\lambda A)^C = \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^\lambda}, (F_A^P)^\lambda\right), \\ \left(-(-T_A^N)^\lambda, -\sqrt{1-(1-(F_A^N)^2)^\lambda}\right) \end{pmatrix}^C$

$= \begin{pmatrix} \left((F_A^P)^\lambda, \sqrt{1-(1-(T_A^P)^2)^\lambda}\right), \\ \left(-\sqrt{1-(1-(F_A^N)^2)^\lambda}, -(-T_A^N)^\lambda\right) \end{pmatrix}$

(3) $A \cup B = \begin{pmatrix} max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\} \end{pmatrix}$

$= \begin{pmatrix} max\{T_B^P, T_A^P\}, min\{F_B^P, F_A^P\}, \\ min\{T_B^N, T_A^N\}, max\{F_B^N, F_A^N\} \end{pmatrix}$

$= B \cup A$

(5) $\lambda(A \cup B) = \lambda \begin{pmatrix} max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\} \end{pmatrix}$

$$= \begin{pmatrix} \left(\sqrt{1-(1-max\{(T_A^P)^2,(T_B^P)^2\})^\lambda}, min\{(F_A^P)^\lambda,(F_B^P)^\lambda\}\right), \\ \left(min\{-(-T_A^N)^\lambda,-(-T_B^N)^\lambda\} - \sqrt{1-(1-(max\{F_A^N,F_B^N\})^2)^\lambda}\right) \end{pmatrix}$$

$$\lambda A \cup \lambda B = \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^\lambda}, (F_A^P)^\lambda\right), \\ \left(-(-T_A^N)^\lambda, -\sqrt{1-(1-(F_A^N)^2)^\lambda}\right) \end{pmatrix} \cup \begin{pmatrix} \left(\sqrt{1-(1-(T_B^P)^2)^\lambda}, (F_B^P)^\lambda\right), \\ \left(-(-T_B^N)^\lambda, -\sqrt{1-(1-(F_B^N)^2)^\lambda}\right) \end{pmatrix}$$

$$= \left(\begin{pmatrix} max\left\{\sqrt{1-(1-(T_A^P)^2)^\lambda}, \sqrt{1-(1-(T_B^P)^2)^\lambda}\right\}, \\ min\{(F_A^P)^\lambda,(F_B^P)^\lambda\} \end{pmatrix}, \begin{pmatrix} min\{-(-T_A^N)^\lambda,-(-T_B^N)^\lambda\}, \\ max\left\{-\sqrt{1-(1-(F_A^N)^2)^\lambda}, -\sqrt{1-(1-(F_B^N)^2)^\lambda}\right\} \end{pmatrix} \right)$$

$$= \begin{pmatrix} \left(\sqrt{1-(1-max\{(T_A^P)^2,(T_B^P)^2\})^\lambda}, min\{(F_A^P)^\lambda,(F_B^P)^\lambda\}\right), \\ \left(min\{-(-T_A^N)^\lambda,-(-T_B^N)^\lambda\}, -\sqrt{1-(1-(max\{F_A^N,F_B^N\})^2)^\lambda}\right) \end{pmatrix}$$

$$= \lambda(A \cup B)$$

(7) Since $T_A^P \geq T_B^P$, $F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}$, $F_A^N \leq F_B^N$.

we have $F_A^P \pi_B^P \leq F_B^P \pi_A^P$

$$(F_A^P)^2(F_B^P)^2 + (F_A^P)^2(\pi_B^P)^2 \leq (F_A^P)^2(F_B^P)^2 + (\pi_A^P)^2(F_B^P)^2$$

$$\Rightarrow \frac{(F_A^P)^2}{(F_B^P)^2} \leq \frac{(F_A^P)^2+(\pi_A^P)^2}{(F_B^P)^2+(\pi_B^P)^2}$$

$$\Rightarrow \left(\frac{(F_A^P)^2}{(F_B^P)^2}\right)^\lambda \leq \left(\frac{(F_A^P)^2+(\pi_A^P)^2}{(F_B^P)^2+(\pi_B^P)^2}\right)^\lambda$$

$$\Rightarrow 1-\left(\frac{(F)^2+(\pi_A^P)^2}{(F_B^P)^2+(\pi_B^P)^2}\right)^\lambda + \left(\frac{(F_A^P)^2}{(F_B^P)^2}\right)^\lambda \leq 1$$

$$\Rightarrow \left(\sqrt{1-\left(\frac{(F_A^P)^2+(\pi_A^P)^2}{(F_B^P)^2+(\pi_B^P)^2}\right)^\lambda}\right)^2 + \left(\frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right)^2 \leq 1$$

$$\Rightarrow \left(\sqrt{1-\left(\frac{1-(T_A^P)^2}{1-(T_B^P)^2}\right)^\lambda}\right)^2 + \left(\frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right)^2 \leq 1.$$

Similarly

$$-\left(\left(\frac{(T_A^N)^\lambda}{(T_B^N)^\lambda}\right)^2 + \left(\sqrt{1-\left(\frac{1-(F_A^N)^2}{1-(F_B^N)^2}\right)^\lambda}\right)^2\right) \geq -1.$$

Then,

$$\lambda(A \ominus B) = \lambda \begin{pmatrix} \left(\sqrt{\frac{(T_A^P)^2-(T_B^P)^2}{1-(T_B^P)^2}}, \frac{F_A^P}{F_B^P}\right), \\ \left(-\frac{T_A^N}{T_B^N}, -\sqrt{\frac{(F_A^N)^2-(F_B^N)^2}{1-(F_B^N)^2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1-\left(1-\frac{(T_A^P)^2-(T_B^P)^2}{1-(T_B^P)^2}\right)^\lambda}, \frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right), \\ \left(-\left(-\left(-\frac{T_A^N}{T_B^N}\right)\right)^\lambda, -\sqrt{1-\left(1-\frac{(F_A^N)^2-(F_B^N)^2}{1-(F_B^N)^2}\right)^\lambda}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1-\left(\frac{1-(T_A^P)^2}{1-(T_B^P)^2}\right)^\lambda}, \frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right), \\ \left(-\frac{(-T_A^N)^\lambda}{(-T_B^N)^\lambda}, -\sqrt{1-\left(\frac{1-(F_A^N)^2}{1-(F_B^N)^2}\right)^\lambda}\right) \end{pmatrix}$$

$\lambda A \ominus \lambda B$

$$= \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^\lambda}, (F_A^P)^\lambda\right), \\ \left(-(-T_A^N)^\lambda, -\sqrt{1-(1-(F_A^N)^2)^\lambda}\right) \end{pmatrix}$$

$$\ominus \begin{pmatrix} \left(\sqrt{1-(1-(T_B^P)^2)^\lambda}, (F_B^P)^\lambda\right), \\ \left(-(-T_B^N)^\lambda, -\sqrt{1-(1-(F_B^N)^2)^\lambda}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{\frac{1-(1-(T_A^P)^2)^\lambda-(1-(1-(T_B^P)^2)^\lambda)}{1-(1-(1-(T_B^P)^2)^\lambda)}}, \frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right), \\ \left(-\frac{(-T_A^N)^\lambda}{(-T_B^N)^\lambda}, -\sqrt{\frac{1-(1-(F_A^N)^2)^\lambda-(1-(1-(F_B^N)^2)^\lambda)}{1-(1-(1-(F_B^N)^2)^\lambda)}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1-\left(\frac{1-(T_A^P)^2}{1-(T_B^P)^2}\right)^\lambda}, \frac{(F_A^P)^\lambda}{(F_B^P)^\lambda}\right), \\ \left(-\frac{(-T_A^N)^\lambda}{(-T_B^N)^\lambda}, -\sqrt{1-\left(\frac{1-(F_A^N)^2}{1-(F_B^N)^2}\right)^\lambda}\right) \end{pmatrix}$$

$= \lambda(A \ominus B)$.

(9) Since $\lambda_1 \geq \lambda_2$, then

$$\lambda_1 A \ominus \lambda_2 A = \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^{\lambda_1}}, (F_A^P)^{\lambda_1}\right), \\ \left(-(-T_A^N)^{\lambda_1}, -\sqrt{1-(1-(F_A^N)^2)^{\lambda_1}}\right) \end{pmatrix} \ominus \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^{\lambda_2}}, (F_A^P)^{\lambda_2}\right), \\ \left(-(-T_A^N)^{\lambda_2}, -\sqrt{1-(1-(F_A^N)^2)^{\lambda_2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{\frac{1-(1-(T_A^P)^2)^{\lambda_1}-(1-(1-(T_A^P)^2)^{\lambda_2})}{1-(1-(1-(T_A^P)^2)^{\lambda_2})}}, \frac{(F_A^P)^{\lambda_1}}{(F_A^P)^{\lambda_2}}\right), \\ \left(-\frac{(-T_A^N)^{\lambda_1}}{(-T_A^N)^{\lambda_2}}, -\sqrt{\frac{1-(1-(F_A^N)^2)^{\lambda_1}-(1-(1-(F_A^N)^2)^{\lambda_2})}{1-(1-(1-(F_A^N)^2)^{\lambda_2})}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1-(1-(T_A^P)^2)^{(\lambda_1-\lambda_2)}}, (F_A^P)^{(\lambda_1-\lambda_2)}\right), \\ \left(-(-T_A^N)^{(\lambda_1-\lambda_2)}, -\sqrt{1-(1-(F_A^N)^2)^{(\lambda_1-\lambda_2)}}\right) \end{pmatrix}$$

$= (\lambda_1 - \lambda_2)A$.

Theorem 3.3

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs, then

(1) $A^C \cup B^C = (A \cap B)^C$;
(2) $A^C \cap B^C = (A \cup B)^C$;
(3) $A^C \oplus B^C = (A \otimes B)^C$;
(4) $A^C \otimes B^C = (A \oplus B)^C$;
(5) $A^C \ominus B^C = (A \oslash B)^C$; $if\ T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}$, $F_A^P \geq F_B^P$, $T_A^N \leq T_B^N$, $F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$
(6) $(A \oslash B) = (A \ominus B)^C$; $if\ T_A^P \geq T_B^P$, $F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,
$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}$, $F_A^N \leq F_B^N$.

.

Proof:

In the following, we shall prove (1),(3),(5) and (2), (4), (6) can be proved similarly.

(1) $A^C \cup B^C (F_A^P, T_A^P, F_A^N, T_A^N) \cup (F_B^P, T_B^P, F_B^N, T_B^N)$

$$= \begin{pmatrix} max\{F_A^P, F_B^P\}, min\{T_A^P, T_B^P\}, \\ min\{F_A^N, F_B^N\}, max\{T_A^N, T_B^N\} \end{pmatrix}$$

$$(A \cap B)^C = \begin{pmatrix} min\{T_A^P, T_B^P\}, max\{F_A^P, F_B^P\}, \\ max\{T_A^N, T_B^N\}, min\{F_A^N, F_B^N\} \end{pmatrix}^C$$

$$= \begin{pmatrix} max\{F_A^P, F_B^P\}, min\{T_A^P, T_B^P\}, \\ min\{F_A^N, F_B^N\}, max\{T_A^N, T_B^N\} \end{pmatrix}$$

$= A^C \cup B^C$

(3) $A^C \oplus B^C = \left((F_A^P, T_A^P), (F_A^N, T_A^N)\right) \oplus \left((F_B^P, T_B^P), (F_B^N, T_B^N)\right)$

$$= \begin{pmatrix} \left(\sqrt{(F_A^P)^2 + (F_B^P)^2 - (F_A^P)^2(F_B^P)^2}, T_A^P T_B^P\right), \\ \left(-F_A^N F_B^N, -\sqrt{(T_A^N)^2 + (T_B^N)^2 - (T_A^N)^2(T_B^N)^2}\right) \end{pmatrix}$$

$$(A \otimes B)^C = \begin{pmatrix} \left(T_A^P T_B^P, \sqrt{(F_A^P)^2 + (F_B^P)^2 - (F_A^P)^2(F_B^P)^2}\right), \\ \left(-\sqrt{(T_A^N)^2 + (T_B^N)^2 - (T_A^N)^2(T_B^N)^2}, -F_A^N F_B^N,\right) \end{pmatrix}^C$$

$$= \begin{pmatrix} \left(\sqrt{(F_A^P)^2 + (F_B^P)^2 - (F_A^P)^2(F_B^P)^2}, T_A^P T_B^P\right), \\ \left(-F_A^N F_B^N, -\sqrt{(T_A^N)^2 + (T_B^N)^2 - (T_A^N)^2(T_B^N)^2}\right) \end{pmatrix}$$

$= A^C \oplus B^C$.

(5) Since $if\ T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}$, $F_A^P \geq F_B^P$

$T_A^N \leq T_B^N$, $F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$,

we have

$A^C \ominus B^C = (F_A^P, T_A^P, F_A^N, T_A^N) \ominus (F_B^P, T_B^P, F_B^N, T_B^N)$

$$= \begin{pmatrix} \left(\sqrt{\frac{(F_A^P)^2 - (F_B^P)^2}{1-(F_B^P)^2}}, \frac{T_A^P}{T_B^P}\right), \\ \left(-\frac{F_A^N}{F_B^N}, -\sqrt{\frac{(T_A^N)^2 - (T_B^N)^2}{1-(T_B^N)^2}}\right) \end{pmatrix}$$

$$(A \oslash B)^C = \begin{pmatrix} \left(\frac{T_A^P}{T_B^P}, \sqrt{\frac{(F_A^P)^2-(F_B^P)^2}{1-(F_B^P)^2}}\right), \\ \left(-\sqrt{\frac{(T_A^N)^2-(T_B^N)^2}{1-(T_B^N)^2}}, -\frac{F_A^N}{F_B^N}\right) \end{pmatrix}^C$$

$$= \begin{pmatrix} \left(\sqrt{\frac{(F_A^P)^2-(F_B^P)^2}{1-(F_B^P)^2}}, \frac{T_A^P}{T_B^P}\right), \\ \left(-\frac{F_A^N}{F_B^N}, -\sqrt{\frac{(T_A^N)^2-(T_B^N)^2}{1-(T_B^N)^2}}\right) \end{pmatrix}$$

$$= A^C \ominus B^C$$

Theorem 3.4

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs, then

(1) $(A \cup B) \oplus (A \cap B) = A \oplus B$;

(2) $(A \cup B) \otimes (A \cap B) = A \otimes B$;

(3) $(A \cup B) \ominus (A \cap B) = A \ominus B$; $if \;\; T_A^P \geq T_B^P, F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}, F_A^N \leq F_B^N$.

(4) $(A \cup B) \oslash (A \cap B) = A \oslash B$; $if \;\; T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}, F_A^P \geq F_B^P$

$T_A^N \leq T_B^N, \quad F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$.

Proof:

In the following, we shall prove (1), (3) and (2), (4) can be proved analogously.

(1) $(A \cup B) \oplus (A \cap B) =$

$$\begin{pmatrix} max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\} \end{pmatrix} \oplus \begin{pmatrix} min\{T_A^P, T_B^P\}, max\{F_A^P, F_B^P\}, \\ max\{T_A^N, T_B^N\}, min\{F_A^N, F_B^N\} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{max\{(T_A^P)^2,(T_B^P)^2\} + min\{(T_A^P)^2,(T_B^P)^2\} - max\{(T_A^P)^2,(T_B^P)^2\} min\{(T_A^P)^2,(T_B^P)^2\}}, \; min\{F_A^P, F_B^P\} max\{F_A^P, F_B^P\}\right), \\ \left(-min\{T_A^N, T_B^N\} \max\{T_A^N, T_B^N\}, \; -\sqrt{(max\{F_A^N, F_B^N\})^2 + (min\{F_A^N, F_B^N\})^2 - (min\{F_A^N, F_B^N\})^2 (max\{F_A^N, F_B^N\})^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{(T_A^P)^2 + (T_B^P)^2 - (T_A^P)^2 (T_B^P)^2}, -F_A^P F_B^P\right), \\ \left(-T_A^N T_B^N, -\sqrt{(F_A^N)^2 + (F_B^N)^2 - (F_A^N)^2 (F_B^N)^2}\right) \end{pmatrix}$$

$$= A \oplus B.$$

(3) Since $if \;\; T_A^P \geq T_B^P, F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}, F_A^N \leq F_B^N$, then

$(A \cup B) \ominus (A \cap B)$

$$= \begin{pmatrix} max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\} \end{pmatrix} \ominus \begin{pmatrix} min\{T_A^P, T_B^P\}, max\{F_A^P, F_B^P\}, \\ max\{T_A^N, T_B^N\}, min\{F_A^N, F_B^N\} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{\frac{max\{(T_A^P)^2,(T_B^P)^2\} - min\{(T_A^P)^2,(T_B^P)^2\}}{1 - min\{(T_A^P)^2,(T_B^P)^2\}}}, \frac{min\{F_A^P, F_B^P\}}{max\{F_A^P, F_B^P\}}\right), \\ \left(-\frac{min\{T_A^N, T_B^N\}}{max\{T_A^N, T_B^N\}}, -\sqrt{\frac{(max\{F_A^N, F_B^N\})^2 - (min\{F_A^N, F_B^N\})^2}{1 - (min\{F_A^N, F_B^N\})^2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{\frac{(T_A^P)^2 - (T_B^P)^2}{1-(T_B^P)^2}}, \frac{F_A^P}{F_B^P}\right), \\ \left(-\frac{T_A^N}{T_B^N}, -\sqrt{\frac{(F_A^N)^2 - (F_B^N)^2}{1-(F_B^N)^2}}\right) \end{pmatrix}$$

$$= A \ominus B$$

Theorem 3.5

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, and $B = (T_B^P, F_B^P, T_B^N, F_B^N)$, be two BPFNs, then

(1) $(A \cup B) \cap B = B$;

(2) $(A \cap B) \cup B = B$;

(3) $(A \ominus B) \oplus B = A$,

$if \;\; T_A^P \geq T_B^P, F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$,

$T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}, F_A^N \leq F_B^N$.

(4) $(A \oslash B) \otimes B = A$,

$if \;\; T_A^P \leq \min\left\{T_B^P, \frac{T_B^P \pi_A^P}{\pi_B^P}\right\}, \; F_A^P \geq F_B^P$

$T_A^N \leq T_B^N, \quad F_A^N \geq \max\left\{F_B^N, \frac{F_B^N \pi_A^N}{\pi_B^N}\right\}$.

Proof:

In the following, we shall Prove (1),(3) and (2),(4) can be proved analogously.

$$(1)(A \cup B) \cap B = \begin{pmatrix} max\{T_A^P, T_B^P\}, \min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\,\{F_A^N, F_B^N\} \end{pmatrix} \cap \left((T_B^P, F_B^P), (T_B^N, F_B^N)\right)$$

$$= \left(\begin{pmatrix} min\{max\{T_A^P, T_B^P\}, T_B^P\}, \\ max\{min\,\{F_A^P, F_B^P\}, F_B^P\} \end{pmatrix}, \begin{pmatrix} max\{min\{T_A^N, T_B^N\}, T_B^N\}, \\ min\{max\,\{F_A^N, F_B^N\}, F_B^N\} \end{pmatrix} \right)$$

$$= (T_B^P, F_B^P, T_B^N, F_B^N) = B.$$

(3) Since $T_A^P \geq T_B^P$, $F_A^P \leq \min\left\{F_B^P, \frac{F_B^P \pi_A^P}{\pi_B^P}\right\}$, $T_A^N \geq \max\left\{T_B^N, \frac{T_B^N \pi_A^N}{\pi_B^N}\right\}$, $F_A^N \leq F_B^N$, then

$$(A \ominus B) \oplus B = \begin{pmatrix} \left(\sqrt{\frac{(T_A^P)^2 - (T_B^P)^2}{1-(T_B^P)^2}}, \frac{F_A^P}{F_B^P}\right), \\ \left(-\frac{T_A^N}{T_B^N}, -\sqrt{\frac{(F_A^N)^2 - (F_B^N)^2}{1-(F_B^N)^2}}\right) \end{pmatrix} \oplus \left((T_B^P, F_B^P), (T_B^N, F_B^N)\right)$$

$$= \left(\begin{pmatrix} \sqrt{\left(\sqrt{\frac{(T_A^P)^2 - (T_B^P)^2}{1-(T_B^P)^2}}\right)^2 + (T_B^P)^2 - \left(\sqrt{\frac{(T_A^P)^2 - (T_B^P)^2}{1-(T_B^P)^2}}\right)^2 (T_B^P)^2}, \\ \frac{F_A^P}{F_B^P} F_B^P \end{pmatrix}, \begin{pmatrix} -\left(-\frac{T_A^N}{T_B^N}\right) T_B^N, \\ -\sqrt{\frac{(F_A^N)^2 - (F_B^N)^2}{1-(F_B^N)^2} + (F_B^N)^2 - \frac{(F_A^N)^2 - (F_B^N)^2}{1-(F_B^N)^2} (F_B^N)^2} \end{pmatrix} \right)$$

$$= (T_A^P, F_A^P, T_A^N, F_A^N) = A.$$

Theorem 3.6

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, $B = (T_B^P, F_B^P, T_B^N, F_B^N)$ and $C = (T_C^P, F_C^P, T_C^N, F_C^N)$ be three BPFNs, then

(1) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
(2) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
(3) $(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$;
(4) $(A \cap B) \oplus C = (A \oplus C) \cap (B \oplus C)$;
(5) $(A \cup B) \otimes C = (A \otimes C) \cup (B \otimes C)$;
(6) $(A \cap B) \otimes C = (A \otimes C) \cap (B \otimes C)$;

Proof:

In the following, we shall prove the (1),(3),(5) and (2),(4),(6) can be proved analogously.

$$(1)(A \cup B) \cap C = \begin{pmatrix} max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}, \\ min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\} \end{pmatrix} \cap \left((T_C^P, F_C^P), (T_C^N, F_C^N)\right)$$

$$= \left(\begin{pmatrix} min\{max\{T_A^P, T_B^P\}, T_C^P\}, \\ max\{min\{F_A^P, F_B^P\}, F_C^P\}, \end{pmatrix}, \begin{pmatrix} max\{min\{T_A^N, T_B^N\}, T_C^N\}, \\ min\{max\{F_A^N, F_B^N\}, F_C^N\} \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} max\{min\{T_A^P, T_C^P\}, min\{T_B^P, T_C^P\}\}, \\ min\{max\{F_A^P, F_C^P\}, max\{F_B^P, F_C^P\}\} \end{pmatrix}, \begin{pmatrix} min\{max\{T_A^N, T_C^N\}, max\{T_B^N, T_C^N\}\}, \\ max\{min\{F_A^N, F_C^N\}, min\{F_B^N, F_C^N\}\} \end{pmatrix} \right)$$

$$= \begin{pmatrix} (min\{T_A^P, T_C^P\}, max\{F_A^P, F_C^P\}), \\ (max\{T_A^N, T_C^N\}, min\{F_A^N, F_C^N\}) \end{pmatrix} \cup \begin{pmatrix} (min\{T_B^P, T_C^P\}, max\{F_B^P, F_C^P\}), \\ max\{T_B^N, T_C^N\}, min\{F_B^N, F_C^N\} \end{pmatrix}$$

$$= (A \cap C) \cup (B \cap C)$$

$$(3)\ (A \cup B) \oplus C = \left((max\{T_A^P, T_B^P\}, min\{F_A^P, F_B^P\}), (min\{T_A^N, T_B^N\}, max\{F_A^N, F_B^N\}\right) \oplus \left((T_C^P, F_C^P), (T_C^N, N_C^N)\right)$$

$$= \left(\begin{pmatrix} \sqrt{max\{(T_A^P)^2, (T_B^P)^2\} + (T_C^P)^2 - max\{(T_A^P)^2, (T_B^P)^2\}(T_C^P)^2}, \\ min\{F_A^P, F_B^P\} F_C^P \end{pmatrix}, \begin{pmatrix} -min\{T_A^N, T_B^N\} T_C^N, \\ -\sqrt{(max\{F_A^N, F_B^N\})^2 + (F_C^N)^2 - (max\{F_A^N, F_B^N\})^2 (F_C^N)^2} \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \sqrt{(1-(T_C^P)^2) max\{(T_A^P)^2, (T_B^P)^2\} + (T_C^P)^2}, \\ min\{F_A^P F_C^P, F_B^P F_C^P\} \end{pmatrix}, \begin{pmatrix} min\{-T_A^N T_C^N, -T_B^N T_C^N\}, \\ \sqrt{(1-(F_C^N)^2)(max\{F_A^N, F_B^N\})^2 + (F_C^N)^2} \end{pmatrix} \right)$$

$$(A \oplus C) \cup (B \oplus C) = \begin{pmatrix} \left(\sqrt{(T_A^P)^2 + (T_C^P)^2 - (T_A^P)^2 (T_C^P)^2}, F_A^P F_C^P\right), \\ \left(-T_A^N T_C^N, -\sqrt{(F_A^N)^2 + (F_C^N)^2 - (F_A^N)^2 (F_C^N)^2}\right) \end{pmatrix} \cup \begin{pmatrix} \left(\sqrt{(T_B^P)^2 + (T_C^P)^2 - (T_B^P)^2 (T_C^P)^2}, F_B^P F_C^P\right), \\ \left(-T_B^N T_C^N, -\sqrt{(F_B^N)^2 + (F_C^N)^2 - (F_B^N)^2 (F_C^N)^2}\right) \end{pmatrix}$$

$$= \left(\left(\begin{matrix} max\left\{\sqrt{(T_A^P)^2+(T_C^P)^2-(T_A^P)^2(T_C^P)^2}, \sqrt{(T_B^P)^2+(T_C^P)^2-(T_B^P)^2(T_C^P)^2}\right\}, \\ min\{F_A^P F_C^P, F_B^P F_C^P\} \end{matrix} \right), \left(\begin{matrix} min\{-T_A^N T_C^N, -T_B^N T_C^N\}, \\ max\left\{-\sqrt{(F_A^N)^2+(F_C^N)^2-(F_A^N)^2(F_C^N)^2}, -\sqrt{(F_B^N)^2+(F_C^N)^2-(F_B^N)^2(F_C^N)^2}\right\} \end{matrix} \right) \right)$$

$$= \left(\left(\begin{matrix} max\left\{\sqrt{(1-(T_C^P)^2)(T_A^P)^2+(T_C^P)^2}, \sqrt{(1-(T_C^P)^2)(T_B^P)^2+(T_C^P)^2}\right\}, \\ min\{F_A^P F_C^P, F_B^P F_C^P\} \end{matrix} \right), \left(\begin{matrix} min\{-T_A^N T_C^N, -T_B^N T_C^N\}, \\ max\left\{-\sqrt{(1-(F_C^N)^2)(F_A^N)^2+(F_C^N)^2}, -\sqrt{(1-(F_C^N)^2)(F_B^N)^2+(F_C^N)^2}\right\} \end{matrix} \right) \right)$$

$$= \left(\left(\begin{matrix} \sqrt{(1-(T_C^P)^2)max\{(T_A^P)^2,(T_B^P)^2\}+(T_C^P)^2}, \\ min\{F_A^P F_B^P, F_A^P F_B^P\} \end{matrix} \right), \left(\begin{matrix} min\{-T_A^N T_C^N, -T_B^N T_C^N\}, \\ -\sqrt{(1-(F_C^N)^2)(max\{F_A^N,F_B^N\})^2+(F_C^N)^2} \end{matrix} \right) \right)$$

$$= (A \oplus C) \cup (B \oplus C).$$

(5) $$(A \cup B) \otimes C = \left(\begin{matrix} (max\{T_A^P,T_B^P\}, min\{F_A^P,F_B^P\}), \\ (min\{T_A^N,T_B^N\}, max\{F_A^N,F_B^N\}) \end{matrix} \right) \otimes (T_C^P,F_C^P,T_C^N,F_C^N)$$

$$= \left(\left(\begin{matrix} max\{T_A^P,T_B^P\}T_C^P, \\ \sqrt{min\{(F_A^P)^2,(F_B^P)^2\}+(F_C^P)^2-min\{(F_A^P)^2,(F_B^P)^2\}(F_C^P)^2} \end{matrix} \right), \left(\begin{matrix} -\sqrt{(min\{T_A^N,T_B^N\})^2+(T_C^N)^2-(min\{T_A^N,T_B^N\})^2(T_C^N)^2}, \\ -max\{T_A^N,T_B^N\}T_C^N \end{matrix} \right) \right)$$

$$= \left(\left(\begin{matrix} max\{T_A^P,T_B^P\}T_C^P, \\ \sqrt{(1-(F_C^P)^2)min\{(F_A^P)^2,(F_B^P)^2\}+(F_C^P)^2} \end{matrix} \right), \left(\begin{matrix} -\sqrt{(1-(T_C^N)^2)(min\{T_A^N,T_B^N\})^2+(T_C^N)^2}, \\ -max\{T_A^N,T_B^N\}T_C^N \end{matrix} \right) \right)$$

$$(A \otimes C) \cup (B \otimes C) = \left(\begin{matrix} \left(T_A^P T_C^P, \sqrt{(F_A^P)^2+(F_C^P)^2-(F_A^P)^2(F_C^P)^2}\right) \\ \left(-\sqrt{(T_A^N)^2+(T_C^N)^2-(T_A^N)^2(T_C^N)^2}, -F_A^N F_C^N\right) \end{matrix} \right) \cup \left(\begin{matrix} \left(T_B^P T_C^P, \sqrt{(F_B^P)^2+(F_C^P)^2-(F_B^P)^2(F_C^P)^2}\right) \\ \left(-\sqrt{(T_B^N)^2+(T_C^N)^2-(T_B^N)^2(T_C^N)^2}, -F_B^N F_C^N\right) \end{matrix} \right)$$

$$= \left(\left(\begin{matrix} max\{T_A^P T_C^P, T_B^P T_C^P\}, \\ min\left\{\sqrt{(F_A^P)^2+(F_C^P)^2-(F_A^P)^2(F_C^P)^2}, \sqrt{(F_B^N)^2+(F_C^N)^2-(F_B^N)^2(F_C^N)^2}\right\} \end{matrix} \right), \left(\begin{matrix} min\left\{-\sqrt{(T_A^N)^2+(T_C^N)^2-(T_A^N)^2(T_C^N)^2}, -\sqrt{(T_B^N)^2+(T_C^N)^2-(T_B^N)^2(T_C^N)^2}\right\} \\ max\{-F_A^N F_C^N, -F_B^N F_C^N\} \end{matrix} \right) \right)$$

$$= \left(\left(\begin{matrix} max\{T_A^P,T_B^P\}T_C^P, \\ \sqrt{(1-(F_C^P)^2)min\{(F_A^P)^2,(F_B^P)^2\}+(F_C^P)^2} \end{matrix} \right), \left(\begin{matrix} -\sqrt{(1-(T_C^N)^2)(min\{T_A^N,T_B^N\})^2+(T_C^N)^2}, \\ -max\{T_A^N,T_B^N\}T_C^N \end{matrix} \right) \right)$$

$$= (A \cup B) \otimes C.$$

Theorem 3.7

Let $A = (T_A^P, F_A^P, T_A^N, F_A^N)$, $B = (T_B^P, F_B^P, T_B^N, F_B^N)$ and $C = (T_C^P, F_C^P, T_C^N, F_C^N)$ be three BPFNs, then

(1) $A \cup B \cup C = A \cup C \cup B$;
(2) $A \cap B \cap C = A \cap C \cap B$;
(3) $A \oplus B \oplus C = A \oplus C \oplus B$;
(4) $A \otimes B \otimes C = A \otimes C \otimes B$;

Proof:

In the following, we shall prove the (1),(3) and (2),(4) can be proved analogously.

(1) $$A \cup B \cup C = (T_A^P, F_A^P, T_A^N, F_A^N) \cup (T_B^P, F_B^P, T_B^N, F_B^N) \cup (T_C^P, F_C^P, T_C^N, F_C^N)$$

$$= \left(\left(\begin{matrix} max\{max\{T_A^P,T_C^P\},T_B^P\}, \\ min\{min\{F_A^P,F_C^P\},F_B^P\} \end{matrix} \right) \left(\begin{matrix} min\{min\{T_A^N,T_C^N\},T_B^N\}, \\ max\{max\{F_A^N,F_C^N\},F_B^N\} \end{matrix} \right) \right)$$

$$= \left(\left(\begin{matrix} max\{max\{T_A^P,T_C^P\},T_B^P\}, \\ min\{min\{F_A^P,F_C^P\},F_B^P\} \end{matrix} \right) \left(\begin{matrix} min\{min\{T_A^N,T_C^N\},T_B^N\}, \\ max\{max\{F_A^N,F_C^N\},F_B^N\} \end{matrix} \right) \right)$$

$$= (T_A^P, F_A^P, T_A^N, F_A^N) \cup (T_C^P, F_C^P, T_C^N, F_C^N) \cup (T_B^P, F_B^P, T_B^N, F_B^N)$$

(3) $$A \oplus B \oplus C = \left(\begin{matrix} \left(\sqrt{(T_A^P)^2+(T_B^P)^2-(T_A^P)^2(T_B^P)^2}, F_A^P F_B^P\right), \\ \left(-T_A^N T_B^N, -\sqrt{(F_A^N)^2+(F_B^N)^2-(F_A^N)^2(F_B^N)^2}\right) \end{matrix} \right) \oplus \left((T_C^P, F_C^P), (T_C^N, F_C^N)\right)$$

$$= \left(\left(\sqrt{(T_A^P)^2+(T_B^P)^2-(T_A^P)^2(T_B^P)^2+(T_C^P)^2-((T_A^P)^2+(T_B^P)^2-(T_A^P)^2(T_B^P)^2)(T_C^P)^2},\ F_A^P F_B^P F_C^P\right), \left(-(-T_A^N T_B^N)T_C^N,\ -\sqrt{(F_A^N)^2+(F_B^N)^2-(F_A^N)^2(F_B^N)^2+(F_C^N)^2-((F_A^N)^2+(F_B^N)^2-(F_A^N)^2(F_B^N)^2)(F_C^N)^2}\right)\right)$$

$$= \left(\left(\sqrt{(T_A^P)^2+(T_C^P)^2-(T_A^P)^2(T_C^P)^2+(T_B^P)^2-((T_A^P)^2+(T_C^P)^2-(T_A^P)^2(T_C^P)^2)(T_B^P)^2},\ F_A^P F_C^P F_B^P\right), \left(-(-T_A^N T_C^N)T_B^N,\ -\sqrt{(F_A^N)^2+(F_C^N)^2-(F_A^N)^2(F_C^N)^2+(F_B^N)^2-((F_A^N)^2+(F_C^N)^2-(F_A^N)^2(F_C^N)^2)(F_B^N)^2}\right)\right)$$

$$= A \oplus C \oplus B.$$

Definition 3.8

Let $a_j = \left(T_j^P, F_j^P, T_j^N, F_j^N\right)$ $(j = 1,2,\dots,n)$ be a family of bipolar Pythagorean fuzzy numbers. A mapping $A_w: \sigma_n \to \sigma$ is called bipolar Pythagorean fuzzy weighted average operator if it satisfies

$$A_w(a_1, a_2, \dots, a_n) = \sum_{j=1}^n a_j{}^{\omega_j}$$
$$\left(\sqrt{1-\prod_{j=1}^n \left(1-\left(T_j^P\right)^2\right)^{\omega_j}}, \prod_{j=1}^n \left(F_j^P\right)^{\omega_j},\right.$$
$$\left.-\prod_{j=1}^n \left(-T_j^N\right)^{\omega_j}, -\sqrt{1-\prod_{j=1}^n \left(1-\left(F_j^N\right)^2\right)^{\omega_j}}\right)$$

where ω_j is the weight of a_j $(j = 1,2,\dots,n)$, $\omega_j \in [0,1]$ and $\sum_{j=1}^n \omega_j = 1$.

Theorem 3.8

Let $a_j = \left(T_j^P, F_j^P, T_j^N, F_j^N\right)$ $(j = 1,2,\dots,n)$ be a family of bipolar Pythagorean fuzzy numbers.

Then,

i. If $a_j = a$ for all $j = 1,2,\dots,n$ then, $A_w(a_1, a_2, \dots, a_n) = a$

ii. $\min_{j=1,2,\dots,n}\{a_j\} \le A_w(a_1, a_2, \dots, a_n) \le \max_{j=1,2,\dots,n}\{a_j\}$

iii. If $a_j \le a'_j$ for all $j = 1,2,\dots,n$ then, $A_w(a_1, a_2, \dots, a_n) \le A_w(a'_1, a'_2, \dots, a'_n)$.

Definition 3.9

Let $a_j = \left(T_j^P, F_j^P, T_j^N, F_j^N\right)$ $(j = 1,2,\dots,n)$ be a family of bipolar Pythagorean fuzzy numbers. A mapping $G_w: \sigma_n \to \sigma$ is called bipolar Pythagorean fuzzy weighted average operator if it satisfies

$$G_w(a_1, a_2, \dots, a_n) = \sum_{j=1}^n a_j{}^{\omega_j} =$$
$$\left(\prod_{j=1}^n \left(T_j^P\right)^{\omega_j}, \sqrt{1-\prod_{j=1}^n \left(1-\left(F_j^P\right)^2\right)^{\omega_j}},\right.$$
$$\left.-\sqrt{1-\prod_{j=1}^n \left(1-\left(T_j^N\right)^2\right)^{\omega_j}}, -\prod_{j=1}^n \left(-F_j^N\right)^{\omega_j}\right)$$

where ω_j is the weight of a_j $(j = 1,2,\dots,n)$, $\omega_j \in [0,1]$ and $\sum_{j=1}^n \omega_j = 1$.

Theorem 3.9

Let $a_j = \left(T_j^P, F_j^P, T_j^N, F_j^N\right)$ $(j = 1,2,\dots,n)$ be a family of bipolar Pythagorean fuzzy numbers.

Then,

i. If $a_j = a$ for all $j = 1,2,\dots,n$ then, $G_w(a_1, a_2, \dots, a_n) = a$

ii. $\min_{j=1,2,\dots,n}\{a_j\} \le G_w(a_1, a_2, \dots, a_n) \le \max_{j=1,2,\dots,n}\{a_j\}$

iii.If $a_j \le a'_j$ for all $j = 1,2,\dots,n$ then, $G_w(a_1, a_2, \dots, a_n) \le G_w(a'_1, a'_2, \dots, a'_n)$.

4.BPFN-DECISION MAKING PROBLEM

In this section, we develop an approach based on the $A_w(or\ G_w)$ operator and the above ranking method to deal with multiple criteria decision making problems with bipolar Pythagorean fuzzy information.

Suppose that $A = \{A_1, A_2, \dots, A_m\}$ and $C = \{C_1, C_2, \dots, C_n\}$ is the set of alternatives and criterions or attributes, respectively. Let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of attributes, such that $\sum_{j=1}^n \omega_j = 1$, $\omega_j \ge 0$ $(j = 1,2,\dots,n)$ and ω_j refers to the weight of attribute C_j.An alternative on criterions is evaluated by the decision maker, and the evaluation values are represented by the form of bipolar Pythagorean fuzzy numbers.

Assume that $\left(a_{ij}\right)_{m\times n} = \left(T_{ij}^P, F_{ij}^P, T_{ij}^N, F_{ij}^N\right)_{m\times n}$.is the decision matrix provided by the decision maker; a_{ij} is a bipolar Pythagorean fuzzy number for alternative A_i associated with the criterions C_j. We have the conditions $T_{ij}^P, F_{ij}^P, T_{ij}^N$ and $F_{ij}^N \in [0,1]$ such that $0 \le \left(T_{ij}^P\right)^2 + \left(F_{ij}^P\right)^2 + \left(T_{ij}^N\right)^2 + \left(F_{ij}^N\right)^2 \le 2$ for $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$.

Now, we can develop an algorithm as follows:

Algorithm

Step 1. Construct the decision matrix provided by the decision maker as:

$$\left(a_{ij}\right)_{m\times n} = \left(T_{ij}^P, F_{ij}^P, T_{ij}^N, F_{ij}^N\right)_{m\times n}.$$

Step2. Compute

$a_i = A_w(a_{i1}, a_{i2}, \dots, a_{in})$ $\left(or\ G_w(a_{i1}, a_{i2}, \dots, a_{in})\right)$ for each $i = 1,2,\dots,m$.

Step 3. Calculate the score values of $S(a_i)$ $(i = 1,2,\dots,m)$ for the collective overall bipolar Pythagorean fuzzy number of a_i $(i = 1,2,\dots,m)$.
Step 4. Rank all the software systems of a_i $(i = 1,2,\dots,m)$ according to the score values.
Now, we give a numerical example as follows:
Example 4.1

A customer who intends to buy a mobile. Four types of mobiles (alternatives) A_i $(i = 1,2,3,4)$ are available.The customer takes into account four attributes to evaluate the alternatives; $C_1 = Storage\ capacity$, $C_2 = Camera\ Pixel\ Quality$, $C_3 = Screen\ Size$, $C_4 = Battery\ Condition$ and use the bipolar Pythagorean fuzzy values to evaluate the four possible alternatives A_i $(i = 1,2,3,4)$ under the above four attributes.Also, the weight vector of the attributes C_j $(j = 1,2,3,4)$ is $\omega = (0.1,0.2,0.3,0.4)^T$.
Then,
Step 1. Construct the decision matrix provided by the customer as:
Table 1: Decision matrix given by customer

Alternatives	C_1	C_2	C_3	C_4
A_1	(0.8,0.2) (-0.4,-0.9)	(0.5,0.3) (-0.8,-0.4)	(0.7,0.5) (-0.7,-0.2)	(0.9,0.4) (-0.7,-0.1)
A_2	(0.5,0.4) (-0.2,-0.7)	(0.9,0.1) (-0.6,-0.6)	(0.5,0.1) (-0.4,-0.6)	(0.8,0.2) (-0.3,-0.4)
A_3	(0.7,0.1) (-0.8,-0.4)	(0.1,0.8) (-0.4,-0.5)	(0.3,0.7) (-0.4,-0.5)	(0.5,0.5) (-0.6,-0.5)
A_4	(0.6,0.5) (-0.2,-0.8)	(0.8,0.5) (-0.5,-0.4)	(0.6,0.3) (-0.1,-0.7)	(0.7,0.2) (-0.2,-0.9)

Step 2. Compute $a_i = A_w(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ for each $i = 1,2,3,4$ as:

$$a_1 = (0.8010, 0.3767, -0.6798, -0.4419)$$
$$a_2 = (0.7584, 0.1516, -0.3607, -0.55)$$
$$a_3 = (0.4377, 0.5173, -0.5042, -0.4914)$$
$$a_4 = (0.6922, 0.2973, -0.1951, -0.7959)$$

Step 3 .Calculate the score values of $S(a_i)$ $(i = 1,2,3,4)$ for the collective overall bipolar Pythagorean fuzzy number of a_i $(i = 1,2,\dots,m)$ as:

$$S(a_1) = 0.3833$$
$$S(a_2) = 0.1899$$
$$S(a_3) = -0.0316$$
$$S(a_4) = -0.1023$$

Step 4. Rank all the software systems of A_i $(i = 1,2,3,4)$ according to the score values as:

$$A_1 > A_2 > A_3 > A_4$$

and thus A_1 is the most desirable alternative.

5. CONCLUSION

In this paper, we have introduced the notions of BPFS, BPFWA operator and BPFGA operator. We have also discussed some of their properties .Finally, a numerical example of the method was given to demonstrate the application.

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