Fibonacci Mean Anti-magic Labeling of Graphs

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Abstract: A Fibonacci Mean Anti-magic labeling of a graph G is an injective function g: V(G) → {f₂, f₃, ... fₙ+1}, where fₙ is nᵗʰ Fibonacci number with the induced a function g*: E(G) → N defined by g*(e = uv) =

\[
\begin{cases} 
\frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\
\frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd}
\end{cases}
\]

and all these edge labeling are distinct. A graph is called Fibonacci Mean Anti-magic if and only if it admits a Fibonacci Mean Anti-magic labeling. In this paper we prove that some special graphs caterpillar, spider, SF(n,1), Bₙₙ, Triangular Snake, Quadrilateral Snake, Ladder and Pₙₙ graphs are Fibonacci Mean Anti-magic graphs.

Keywords: Fibonacci mean anti-magic graph, caterpillar, spider, SF(n,1), Triangular Snake, Quadrilateral Snake, Ladder and Pₙₙ graphs

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1. INTRODUCTION

In this paper we mean a graph G by finite, connected, undirected graph G = (V, E) without any loops and multiple edges with |V| = p vertices and |E| = q edges. We follow the notation and terminology of [9]. The concept of graph labeling was introduced by Rosa [6] in 1967.

Definition 1.1

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The process of vertex labeling is label the vertices with integers. Under this vertex labeling, the edge weight of an edge e = uv is defined as W(e) = g(u) + g(v). In 1994, N.Hartsfield and Ringel [1] introduced the concept of anti-magic graph.

Definition 1.2

Each vertex labeling f of a graph G = (p,q) from{0,1,2,...q} induces a edge labeling fₑ where g*(e) is sum the labels of end vertices of an edge e. Labeling f is called anti-magic if and only if all the edge labelings are pair wisely distinct.

Definition 1.3

By an edge anti-magic vertex labeling we mean a one-to-one mapping V(G) into {0,1,2,...q} such that the set of edge weights of all edges in G is {1,2,...q}. Different kinds of anti-magic graphs were studied by T.Nicholas, S.Somasundaram and V.Vilfred [5]. S.Somasundaram and R.Ponraj [7] and [8] introduced the concept of mean labeling.

Definition 1.4

A graph G with p vertices and q edges is a mean graph if there is an injective function f from the vertices of G to {1,2,...q} such that when each edge uv is labelled with \(\frac{f(u) + f(v)}{2}\) if f(u) + f(v) is even and \(\frac{f(u) + f(v) + 1}{2}\) if f(u) + f(v) is odd then the resulting edges are distinct.

Fibonacci graceful labeling was introduced by Kathiresan.M and Amutha.S [2] and different kind of graphs were studied by [3] and [4].

Definition 1.5

The Fibonacci numbers can be defined by the linear recurrence relation \(F_n = F_{n-1} + F_{n-2}\); n ≥ 3. This generates the infinite sequence of integer’s beginning1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

Definition 1.6

A Caterpillar is a tree which has path \(P_n = a_1\), \(a_2\), ... \(a_n\) of order n and is obtained by attaching \(X_i\) (possibly zero) end vertices at the vertex \(a_i\) of \(P_n\), i = 1,2,...n by end edges. It is denoted as \(T = S(X_1, X_2..., X_n)\). The order of T is n + ∑Xi.
Example 1.7

\[ T = S(X_1, X_2, \ldots, X_n) \]

Fig (1.1) - \[ T = S(X_1, X_2, \ldots, X_n) \]

Definition 1.8

A Spider \( SP(P_n, 2) \) is a Caterpillar \( S(X_1, X_2, \ldots, X_n) \) where \( X_n = 2 \) and \( X_i = 0 \) for \( i = 1, 2, \ldots, n-1 \)

Example 1.9

The following graph is the example for the Spider \( SP(P_4, 2) \)

Fig (1.2) - \( SP(P_4, 2) \)

Definition 1.10

A Quadrilateral Snake \( Q_n \) is obtained from a path \( \{u_1, u_2, \ldots, u_n\} \) by joining \( u_i \) and \( u_{i+1} \) to two vertices \( v_i \) and \( w_i \), \( 1 \leq i \leq n - 1 \) respectively and then joining \( v_1 \) and \( w_n \).

Definition 1.11

The Product \( P_2 \times P_n \) is called a Ladder and it is denoted by \( L_n \).

Definition 1.12

Triangular Snake \( T_n \) is obtained from a path \( u_1, u_2, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) to a new vertex \( v_i \) for \( 1 \leq i \leq n - 1 \), that is every edge of a path is replaced by a triangle \( C_3 \).

Definition 1.13

An \( SF(n,m) \) is a graph consisting of a cycle \( C_n \), \( n \geq 3 \), and \( n \) set of \( m \) independent vertices where each set joins each of the vertices of \( C_n \).

Definition 1.14

Bi-Star is the graph obtained by joining the apex vertices of two copies of Star \( K_{1,n} \).

Definition 1.15

The graph \( P_{v(m)} = G( V, E ) \) such that \( V(G) = \{ v_{ij} | 1 \leq i \leq m, \text{ and } 1 \leq j \leq n \} \)
\[ E(G) = \{ v_{ij}v_{i(j+1)} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1 \} \cup \{ v_{i(m-1)}v_{(i+1)(m-1)} | 1 \leq i \leq m - 1 \} \]

2. RESULTS ON FIBONACCI MEAN ANTI-MAGIC LABELING

In this section we investigate Fibonacci Mean Anti-magic labeling of some special graphs.

Definition 2.1

A Fibonacci Mean Anti-magic labeling of a graph \( G \) is an injective function \( g: V(G) \rightarrow \{ f_2, f_3, \ldots, f_{n+1} \} \), where \( f_n \) is \( n^{th} \) Fibonacci number with the induced a function \( g^* : E(G) \rightarrow N \) defined by
\[ g^*(e = uv) = \begin{cases} \frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\ \frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd} \end{cases} \]

all these edge labeling are distinct. A graph is called Fibonacci Mean Anti-magic if it admits a Fibonacci Mean Anti-magic labeling.

Theorem 2.2

The Caterpillar \( S(X_1, X_2, \ldots, X_n) \) has Fibonacci Mean – Anti-magic labeling.

Proof:

The path vertices are denoted as \( v_1, v_2, \ldots, v_n \) and the end vertices are denoted as \( u_1, u_2, \ldots, u_n \).

The assignment of vertex labels are \( g(v_i) = f_2, f_3, \ldots, f_{n+1} \). The induced edge labels are \( g^*: E(G) \rightarrow N \) and all are distinct. This completes the proof.

Theorem 2.3

\[ \]
Every Spider $SP(P_{n,2})$ admits Fibonacci Mean – Anti-magic labeling.

**Proof:**
Define $g: V(G) \rightarrow \{ f_{1}, f_{3}, \ldots, f_{n+1} \}$, where $f_{i}$ is $n^{th}$ Fibonacci number with the induced a function $g^*: E(G) \rightarrow N$ defined by

\[
g^* (e = uv ) = \begin{cases} 
\frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\
\frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd}
\end{cases}
\]

and all these edge labeling are distinct.

**Example 2.4**
The following graph is the example for the Spider $SP(P_{4,2})$

![Graph Example](image)

**Theorem 2.5**
The graph $G \odot K_{1}$ is a Fibonacci Mean – Anti-magic graph where $G = T_{n}$ for all integer $n \geq 2$.

**Proof:**
Let $\{u_{1}, u_{2}, \ldots, u_{n}\}$ be a path of length $n$. Let $v_{i}, 1 \leq i \leq n - 1$ be the new vertex joined to $u_{i}$ and $u_{i+1}$. The resulting graph is called $T_{n}$. Now the vertex set of $V(G_{1}) = \{u_{1}, u_{2}, \ldots, u_{n}, v_{1}, v_{2}, \ldots, v_{n-1}, w_{1}, w_{2}, \ldots, w_{n-1}, x_{1}, x_{2}, \ldots, x_{n}, y_{1}, y_{2}, \ldots, y_{n}\}$ and the edge set $E(G_{1}) = \{u_{1}u_{2}, u_{1}v_{1}, v_{1}v_{2}, \ldots, v_{n-1}v_{n}, w_{1}w_{2}, \ldots, w_{n-1}w_{n}, x_{1}x_{2}, \ldots, x_{n}, y_{1}y_{2}, \ldots, y_{n}\}$ and the edge set $E(G_{2}) = \{u_{1}u_{i+1}, u_{i}v_{i}/1 \leq i \leq n-1\} \cup \{u_{i}v_{i}, 1 \leq i \leq n\} \cup \{u_{i}v_{i}/1 \leq i \leq n-1\}$. Here $|V(G_{1})| = 4n - 2$.

Let $g: V(G) \rightarrow \{ f_{2}, f_{3}, \ldots, f_{n+1}\}$, where $f_{i}$ is $n^{th}$ Fibonacci number with the induced a function $g^*: E(G) \rightarrow N$ defined by

\[
g^* (e = uv ) = \begin{cases} 
\frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\
\frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd}
\end{cases}
\]

all these edge labeling are distinct.

**Theorem 2.6**
The graph $G \odot K_{1}$ is a Fibonacci Mean – Anti-magic graph where $G = L_{n}$ for all integer $n \geq 2$.

**Proof:**
Let $\{u_{1}, u_{2}, \ldots, u_{n}\}$ be a path of length $n$. Let $v_{i}, w_{i}$ be two vertices joined to $u_{i}$ and $u_{i+1}$ respectively and then join $v_{i}$ and $w_{i}, 1 \leq i \leq n - 1$. The resulting graph is called a quadrilateral snake $Q_{n}$. Let $x_{i}$ be the vertex which is joined to $u_{i}, 1 \leq i \leq n$, let $y_{i}$ be the new vertex which is joined to $v_{i}, 1 \leq i \leq n - 1$ and let $z_{i}$ be the new vertex which is joined to $w_{i}, 1 \leq i \leq n - 1$. The resulting graph is $G_{1} = G \odot K_{1}$ where $G = Q_{n}$. Now the vertex set $V(G_{1}) = \{u_{1}, u_{2}, \ldots, u_{n}, v_{1}, v_{2}, \ldots, v_{n-1}, w_{1}, w_{2}, \ldots, w_{n-1}, x_{1}, x_{2}, \ldots, x_{n}, y_{1}, y_{2}, \ldots, y_{n}\}$ and the edge set $E(G_{1}) = \{u_{1}u_{2}, u_{1}v_{1}, v_{1}v_{2}, \ldots, v_{n-1}v_{n}, w_{1}w_{2}, \ldots, w_{n-1}w_{n}, x_{1}x_{2}, \ldots, x_{n}, y_{1}y_{2}, \ldots, y_{n}\}$ and the edge set $E(G_{2}) = \{u_{1}u_{i+1}, u_{i}v_{i}/1 \leq i \leq n-1\} \cup \{u_{i}v_{i}, 1 \leq i \leq n\} \cup \{u_{i}v_{i}/1 \leq i \leq n-1\}$. Here $|V(G_{1})| = 6n - 4$.

Let $g: V(G) \rightarrow \{ f_{2}, f_{3}, \ldots, f_{n+1}\}$, where $f_{i}$ is $n^{th}$ Fibonacci number with the induced a function $g^*: E(G) \rightarrow N$ defined by

\[
g^* (e = uv ) = \begin{cases} 
\frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\
\frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd}
\end{cases}
\]

all these edge labeling are distinct.
\[ g(e = uv) = \begin{cases} \frac{g(u) + g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\ \frac{g(u) + g(v) + 1}{2} & \text{if } g(u) + g(v) \text{ is odd} \end{cases} \]

These edge labeling are distinct.

**Theorem 2.8**
The graph \( SF(n, 1) \) admits a Fibonacci Mean Anti-magic labeling.

**Proof:**
Let \( G \) denote the graph \( SF(n, 1) \).
Let \( v_1, v_2, \ldots, v_n \) be the vertices of the cycle \( SF(n,1) \) and \( v_j \) for \( j = 1,2,\ldots,n \) be the vertices joining the corresponding vertices \( v_j \).
Here \( |V(G)| = 2n \) and \( |E(G)| = 2n \).
Define \( g: V(G) \to \{ f_2, f_3, \ldots, f_{n+1} \} \), where \( f_n \) is the \( n \)-th Fibonacci number.

The distinct edge labels are determined by the condition
\[ g^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \]
Hence Proved.

**Theorem 2.9**
The Bi- Star \( B_{n,n} \) admits Fibonacci Mean Anti-magic Labeling.

**Proof:**
Consider the two copies of \( K_{1,n} \). Let \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_n \) be the corresponding vertices of each copy of \( K_{1,n} \) with apex vertex \( v \) and \( u \).
Let \( e_i = v v_i \), \( e_i = u u_i \) and \( e = uv \) of bistar graph.
Here \( |V(B_{n,n})| = 2n + 2 \), \( |E(B_{n,n})| = 2n + 1 \).
The vertices are assigned by Fibonacci numbers and the induced edge labels are distinct and the theorem follows.

**Theorem 2.10**
The graph \( P_{n(m)} \) is a Fibonacci Mean Anti-magic graph for all \( n, m \geq 2 \).

**Proof:**
Let \( G = P_{n(m)} \).
Let \( V(G) = \{ v_{ij} | 1 \leq i \leq m, \text{ and } 1 \leq j \leq n \} \) and \( E(G) = \{ v_{ij}v_{i(j+1)} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1 \} \cup \{ v_{i(n-1)}v_{i+1(n-1)} | 1 \leq i \leq m - 1 \} \).
Then \( |V(G)| = mn \) and \( |E(G)| = mn - 1 \).
The assignment of vertex labels are \( g(v_i) = f_2, f_3, \ldots, f_{n+1} \).
All the edge labels are distinct.
Hence the theorem follows.

REFERENCES