New Class of Generalized Closed Sets in Supra Topological Spaces

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Abstract- In this paper we introduce supra $\overline{g}_\alpha$-closed sets and we show that the class of supra $\overline{g}_\alpha$-closed sets lies between the class of supra $\alpha$-closed sets and the class of supra generalized semi-closed sets. We also introduce and discuss some properties of $T_{\beta\overline{g}_\alpha}$-spaces, $\overline{g}\beta_{\overline{g}_\alpha}$-spaces, $\#_{\overline{g}_\alpha}$-spaces. Further we define and investigate the properties of supra-$\overline{g}_\alpha$-continuous functions.

Index Terms- Supra $\alpha$-closed sets, Supra gs-closed sets, Supra $\overline{g}_\alpha$-closed sets, Supra-$\overline{g}_\alpha$-continuous function.

1. INTRODUCTION

Extensive research on generalizing closedness was done in recent years by many Mathematicians. In 1983, A.S.Mashhour et al.[17] introduced the concept of supra topological spaces. M.Caldas et al.[5] introduced and studied supra $\alpha$-open sets and its continuity. In 2010, O.R.Sayed [20] concentrated on supra pre-open sets and supra pre-continuity on topological spaces. In 1970, Levine.N [13] introduced and established the properties of generalized closed sets in classical topology. Many researchers [ 1, 2, 4, 6, 18] turned their attention to define new sets and functions in topological spaces. In 1980, Levine and Dunham [7] further characterized some more properties of generalized closed sets. Noiri et al. [15] proved that every topological space is pre-$T_2$-space. Further some spaces are derived by Maheshwari and Prasad [16]. Recently Jayaparthasarathy, Hydar Akca and Jamal Benbouranene [8, 9] defined and investigated some properties of the new class of generalized closed sets namely $\overline{g}_\alpha$-closed sets. In this paper we define and discuss a new class of weakly $\alpha$-generalized closed set in supra topological spaces called supra $\overline{g}_\alpha$-closed set and investigate its properties. We observe that the class of supra $\overline{g}_\alpha$-closed sets lies between the class of supra $\alpha$-closed sets and the class of supra generalized semi-closed sets. By applying supra $\overline{g}_\alpha$-closed sets, we introduce and establish some properties of $T_{\beta\overline{g}_\alpha}$-spaces, $\overline{g}\beta_{\overline{g}_\alpha}$-spaces. Moreover the supra $\overline{g}_\alpha$-continuous functions are defined and its properties are investigated.

2. PRELIMINARIES

In this section we recall some basic definitions and properties of supra topology which are useful in sequel.

Definition 2.1 Let X be a non-empty set. A sub collection $\mu$ of $P(X)$ where $P(X)$ denote the power set of X, is said to be a supra topology on X [17] if

(i) $\emptyset, X \in \mu$
(ii) $\mu$ is closed under arbitrary union.

Definition 2.2 Let $(X, \mu)$ be a supra topological space and $A \subseteq X$. Then

(i) The supra closure of A is denoted by $cl^\mu(A)$, defined as $cl^\mu(A)=$ $\bigcap \{B:B \subseteq X \text{ and } A \subseteq B \}$. The pair $(X, \mu)$ is called supra topological space. The element of $\mu$ are called the supra open sets in $(X, \mu)$ and the complement of the supra open sets are called supra closed sets.

Definition 2.3 Let $(X, \tau)$ be a topological space and $\mu$ be a supra topology on X. We call $\mu$ a supra topology associated with $\tau$ [17] if $\tau$ a subset of $\mu$. Then

(i) $\mu$ is closed under arbitrary union.

Definition 2.4 Let $(X, \mu)$ be a supra topological space. A subset $A$ of X is called

i) supra $\alpha$-open [14] if $A \subseteq int^\mu(cl^\mu(int^\mu(A)))$.
ii) supra semi-open [12] if $A \subseteq cl^\mu(int^\mu(A))$.
iii) supra pre-open [3] if $A \subseteq int^\mu(cl^\mu(A))$.

The complement of a supra $\alpha$-open set (resp. supra semi-open, and supra pre-open) is called supra $\alpha$-closed (resp. supra semi-closed, and supra pre-closed).

Theorem 2.5 (i) Every supra open set (supra closed set) is supra $\alpha$-open set (supra $\alpha$-closed set) [5].
(ii) Let $(X, \tau)$ be a topological space and $\mu$ be a supra topology associated with $\tau$. Then a subset $A$ of X is supra $\alpha$-open (supra $\alpha$-closed) set if and only if it is supra semi-open (supra semi-closed) set and supra pre-open (supra pre-closed) set [5, 20].

Definition 2.6 A subset $A$ of a supra topological space $(X, \mu)$ is called

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called a topology with \( \tau \). A function \( f: (X, \tau) \to (Y, \sigma) \) is

**Definition 2.7**

The complement of the above mentioned sets are closed set in \( Y \) is a continuous map if the inverse image of each closed set in \( Y \) is a semi-open set in \( (X, \mu) \).

**Remark 3.3**

The converse of the above theorem need not be true and it is shown by the following example.

**Example 3.4**

Let \( X = \{a, b, c, d\} \), \( \mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), then the set \( \{a\} \) is \( \tilde{\varphi}_a^\mu \)-closed but it is not supra closed.

**Theorem 3.6**

Every supra closed set is \( \tilde{\varphi}_a^\mu \)-closed set.

**Proof.** Let A be a supra \( \alpha \)-closed and U be any supra \( \tilde{s}^\mu \)-open set containing A, then \( acl^\mu(A) = \varnothing \subseteq U \). Hence A is \( \tilde{\varphi}_a^\mu \)-closed.

**Theorem 3.8**

Every \( \tilde{\varphi}_a^\mu \)-closed set is \( \tilde{\varphi}_a^\mu \)-closed set but not conversely.

**Proof.** Let A be any \( \tilde{\varphi}_a^\mu \)-closed set and U be any \( \tilde{s}^\mu \)-open set containing A. Since every \( \tilde{s}^\mu \)-open set is \( \tilde{\varphi}_a^\mu \)-open set [11], we have \( \varnothing \subseteq U \) and \( acl^\mu(A) \subseteq cl^\mu(A) \subseteq U \). Hence A is \( \tilde{\varphi}_a^\mu \)-closed.

**Example 3.9**

Let \( X = \{a, b, c, \} \), \( \mu = \{\emptyset, \{a\}, \{b\}, \} \), \( X \). Then the set \( \{c\} \) is \( \tilde{\varphi}_a^\mu \)-closed but not \( \tilde{\varphi}_a^\mu \)-closed.

**Theorem 3.10**

Every \( \tilde{\varphi}_a^\mu \)-closed set is \( \tilde{s}^\mu \)-closed.

**Proof.** Let A be a \( \tilde{\varphi}_a^\mu \)-closed set and U be any supra open set containing A. Since every supra open set is \( \tilde{\varphi}_a^\mu \)-open, we have \( acl^\mu(A) \subseteq \varnothing \subseteq U \). Hence A is \( \tilde{s}^\mu \)-closed.

**Remark 3.11**

The converse of the above theorem need not be true as shown in the following example.

**Example 3.12**

Let \( X = \{a, b, c\} \), \( \mu = \{\emptyset, \{a\}, \} \), \( X \). Then the set \( \{a\} \) is \( \tilde{\varphi}_a^\mu \)-closed but not \( \tilde{\varphi}_a^\mu \)-closed.

**Theorem 3.13**

Every \( \tilde{\varphi}_a^\mu \)-closed set is \( \tilde{s}^\mu \)-closed set but not conversely.

**Proof.** Let A be a \( \tilde{\varphi}_a^\mu \)-closed set and U be any supra open set containing A. Since every supra open set is \( \tilde{s}^\mu \)-open, we have \( acl^\mu(A) \subseteq \varnothing \subseteq U \). Hence A is \( \tilde{s}^\mu \)-closed.

**Proposition 3.15**

Every \( \tilde{\varphi}_a^\mu \)-closed set is \( \tilde{s}^\mu \)-closed set but not conversely.
Proof: Let $A$ be a $\bar{g}_\alpha$-closed set and $U$ be any $\bar{g}s^\mu$-closed set containing $A$, we have $acl^\mu(A) \subseteq acl^\mu(A) \subseteq U$. Hence $A$ is $\bar{g}s^\mu$-closed.

Example 3.16 Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the set $\{b, c\}$ is $\bar{g}s^\mu$-closed set but not $\bar{g}_\alpha$-closed.

Remark 3.17 Here we observe that the class of $\bar{g}_\alpha$-closed sets is properly placed between the class of supra $\alpha$-closed sets and the class of $\bar{g}s^\mu$-closed sets and between the class of supra $\alpha$-closed sets and the class of $\bar{g}s^\mu$-closed sets.

Remark 3.18 The following examples shows that $\bar{g}_\alpha$-closed set is independent from supra- semi-closed set, $g^\mu$-closed set.

Example 3.19 (i) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, X\}$. Then the set $\{a, b\}$ is supra-g-closed but not $\bar{g}_\alpha$-closed.

(ii) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the set $\{a, b\}$ is $\bar{g}_\alpha$-closed but not supra-g-closed.

Definition 3.20 A subset $X$ of $\bar{g}_\alpha$-closed is $\bar{g}_\alpha$-open if and only if $A'$ is $\bar{g}_\alpha$-closed.

Theorem 3.21 (i) Every supra-open set is $\bar{g}_\alpha$-open but not conversely.

(ii) Every supra-$\alpha$-open set is $\bar{g}_\alpha$-open but not conversely.

(iii) Every $\mu$-open set is $\bar{g}_\alpha$-open but not conversely.

(iv) Every $\bar{g}_\alpha$-open set is supra-gs-open, supra- $g^\mu$-open but not conversely.

(v) Every $\bar{g}_\alpha$-open set is supra-ag-open, supra- $g^\mu$-open but not conversely.

(vi) Every $\bar{g}_\alpha$-open set is $\bar{g}s^\mu$-open set but not conversely.

4. CHARACTERIZATION OF SUPRA $\bar{g}_\alpha$-CLOSED SETS

In this section by applying $\bar{g}_\alpha$-closed sets, some new classes of spaces and its properties are investigated.

Theorem 4.1 A set $A$ is $\bar{g}_\alpha$-closed set of $(X, \mu)$, then $acl^\mu(A) \setminus A$ does not contain any non empty $\bar{g}s^\mu$-closed set.

Proof. Suppose that $A$ is $\bar{g}_\alpha$-closed in $(X, \mu)$. Let $F$ be a $\bar{g}s^\mu$-closed subset of $acl^\mu(A) \setminus A$. Now $F'$ is $\bar{g}s^\mu$-open set of $(X, \mu)$ such that $A \subseteq F'$. Since $A$ is $\bar{g}_\alpha$-closed set, $acl^\mu(A) \subseteq F'$, that is, $F \subseteq acl^\mu(A)$. Thus $F \subseteq acl^\mu(A) \cap acl^\mu(A) = \emptyset$. Hence $F$ is empty.

Theorem 4.2 If $A$ is $\bar{g}s^\mu$-open and $\bar{g}_\alpha$-closed subset of $(X, \mu)$, then $A$ is supra $\alpha$-closed.

Proof. Since $A$ is $\bar{g}s^\mu$-open and $\bar{g}_\alpha$-closed, $acl^\mu(A) \subseteq A$. But $A \subseteq acl^\mu(A)$. Thus $acl^\mu(A) = A$, hence $A$ is supra $\alpha$-closed.

Theorem 4.3 If $A$ is a $\bar{g}_\alpha$-closed set in supra topological space $(X, \mu)$ and $A \subseteq B \subseteq acl^\mu(A)$ then $B$ is also $\bar{g}_\alpha$-closed.

Proof. Let $U$ be any $\bar{g}s^\mu$-open set such that $B \subseteq U$. Since $A \subseteq B$ implies $A \subseteq U$. Since $A$ is $\bar{g}_\alpha$-closed then $acl^\mu(A) \subseteq U$. Since $B \subseteq acl^\mu(A)$, $acl^\mu(B) \subseteq acl^\mu(A) \subseteq U$. Thus $acl^\mu(B) \subseteq U$ and hence $B$ is $\bar{g}_\alpha$-closed.

Theorem 4.4 Let $A$ be $\bar{g}_\alpha$-closed set in supra topological space $(X, \mu)$, then $A$ is supra $\alpha$-closed subset of $(X, \mu)$ if and only if $acl^\mu(A) \setminus A$ is $\bar{g}s^\mu$-closed.

Proof. Let $A$ be $\bar{g}_\alpha$-closed set. If $A$ is supra $\alpha$-closed subset of $(X, \mu)$. Then $acl^\mu(A) = A$ and so $acl^\mu(A) \setminus A = \emptyset$. Conversely, let $B$ is $\bar{g}s^\mu$-closed. Then $B$ is $\bar{g}_\alpha$-closed by Theorem 4.1. So $acl^\mu(A) \setminus A$ does not contain any non-empty $\bar{g}s^\mu$-closed set, then $acl^\mu(A) \setminus A = \emptyset$. That is $acl^\mu(A) = A$. Hence $A$ is supra $\alpha$-closed in $(X, \mu)$.

Theorem 4.5 The intersection of a $\bar{g}_\alpha$-closed set and a supra- $\alpha$-closed set is always $\bar{g}_\alpha$-closed set in $(X, \mu)$.

Proof. Let $A$ be $\bar{g}_\alpha$-closed set and let $B$ be supra- $\alpha$-closed set. If $G$ is an $\bar{g}s^\mu$-open set with $A \cap F \subseteq G$, then $A \subseteq G \cap F$. Then $acl^\mu(A \cap F) \subseteq acl^\mu(A) \cap F \subseteq G$. Hence $A \cap F$ is $\bar{g}_\alpha$-closed.

Theorem 4.6 A set $A$ is $\bar{g}_\alpha$-open in $(X, \mu)$ if and only if $F \subseteq acl^\mu(A)$ whenever $F$ is $\bar{g}s^\mu$-closed and $F \subseteq A$.

Proof. Let $A$ be $\bar{g}_\alpha$-open set and suppose $F \subseteq A$ where $F$ is $\bar{g}s^\mu$-closed set. Then $X - A$ is $\bar{g}_\alpha$-closed set contained in the $\bar{g}s^\mu$-open set $X - F$. Hence $acl^\mu(X - A) \subseteq X - F$. Thus $F \subseteq acl^\mu(A)$. Conversely, if $F$ is a $\bar{g}s^\mu$-closed with $F \subseteq acl^\mu(A)$ and $F \subseteq A$, then $X - acl^\mu(A) \subseteq X - F$. This implies that $acl^\mu(X - A) \subseteq X - F$. Hence $A$ is $\bar{g}_\alpha$-closed. Thus $A$ is $\bar{g}_\alpha$-open set in $(X, \mu)$.

Theorem 4.7 If $acl^\mu(A) \subseteq B \subseteq A$ and $A$ is $\bar{g}_\alpha$-open then $B$ is $\bar{g}_\alpha$-open.

Proof. By hypothesis $(A')^c \subseteq (B')^c \subseteq acl^\mu(A')$. That is, $(A')^c \subseteq B' \subseteq acl^\mu(A') \subseteq acl^\mu(A')$. Since $A'$ is $\bar{g}_\alpha$-closed and by Theorem 4.3, $B'$ is $\bar{g}_\alpha$-closed. Hence $B$ is $\bar{g}_\alpha$-open.

Theorem 4.8 A set $A$ is $\bar{g}_\alpha$-open in $(X, \mu)$ if and only if $G = X$, where $G$ is $\bar{g}s^\mu$-open set and $acl^\mu(A) \cup A' \subseteq G$.

Proof. Let $A$ be a $\bar{g}_\alpha$-open set and $G$ be $\bar{g}s^\mu$-open and $acl^\mu(A) \cup A' \subseteq G$. This gives $G' \subseteq acl^\mu(A) \cup A' \subseteq acl^\mu(A) \subseteq acl^\mu(A') \subseteq acl^\mu(A')$. Since $A'$ is $\bar{g}_\alpha$-closed and $G'$ is $\bar{g}s^\mu$-closed. Then $G' = \emptyset$. Thus $G = X$. Conversely, if $F$ is $\bar{g}s^\mu$-closed and $F \subseteq A$. Then $acl^\mu(A) \cup A' \subseteq acl^\mu(A) \cup F$. It follows by hypothesis that $acl^\mu(A) \cup F = X$ and hence $F \subseteq acl^\mu(A)$. Therefore by Theorem 4.6, $A$ is $\bar{g}_\alpha$-open in $(X, \mu)$.
Proof. If \( \{x\} \) is not \( gs^\mu \)-closed set in \((X, \mu)\), then \( \{x\} \) is not \( gs^\mu \)-open and the only \( gs^\mu \)-open set containing \( \{x\} \) is the space \( X \) itself. Therefore \( acl^P(\{x\}) \subseteq X \) and so \( \{x\} \) is \( g_a^\mu \)-closed set in \((X, \mu)\).

Definition 4.10 A space \((X, \mu)\) is called a \( T^\mu \) space if every \( g_a^\mu \)-closed set is supra closed.

Theorem 4.11 For a supra topological space \((X, \mu)\), the following are equivalent:

(i) \( X \) is a \( T^\mu \) space.

(ii) Every singleton of \( X \) is either \( gs^\mu \)-closed or supra \( \alpha \)-open.

Proof. \( i \Rightarrow ii \): Let \( x \in X \). Suppose \( \{x\} \) is not \( gs^\mu \)-closed, then \( X \setminus \{x\} \) is not \( gs^\mu \)-open. Since \( X \) is the only \( gs^\mu \)-open set containing \( X \setminus \{x\} \). Thus \( X \setminus \{x\} \) is \( g_a^\mu \)-closed set in \((X, \mu)\). Since \( X \) is \( T^\mu \) space, \( X \setminus \{x\} \) is supra \( \alpha \)-closed set of \((X, \mu)\), that is \( \{x\} \) is supra \( \alpha \)-open.

Proof. \( ii \Rightarrow i \): Let \( x \in A \) belong to a \( g_a^\mu \)-closed set. Let \( x \in acl^P(A) \). By \( ii \), \( \{x\} \) is either \( gs^\mu \)-closed or supra \( \alpha \)-open.

Case (i): If \( \{x\} \) is \( gs^\mu \)-closed. We assume that \( x \in A \), then \( acl^P(A) \setminus A \) contains the \( gs^\mu \)-closed set \( \{x\} \) and \( A \) is \( g_a^\mu \)-closed set, which cannot happen according to Theorem 4.1. Hence \( x \notin A \).

Case (ii): If \( \{x\} \) is supra \( \alpha \)-open, since \( x \in acl^P(A) \) then \( X \setminus \{x\} \cap A = \phi \). This shows that \( x \notin A \). Thus in both case we have \( acl^P(A) \subseteq A \). Therefore \( A = acl^P(A) \). That is, \( A \) is supra \( \alpha \)-closed. Thus \( X \) is an \( T^\mu \) space.

Definition 4.12 A space \((X, \mu)\) is called a \( \#_g^\mu \) space if every \( g_a^\mu \)-closed set is supra closed.

Theorem 4.13 Every \( \#_g^\mu \) space is \( T^\mu \) space, but converse not true.

Proof. Since every supra closed set is supra \( \alpha \)-closed, the proof follows immediately.

Example 4.14 Let \( X = \{a, b, c\} \); \( \mu = \{\emptyset, \{a\}, \{b, c\}\} \). Here \( X \) is an \( \#_g^\mu \) space but not \( \#_g^\mu \) space.

Theorem 4.15 If \((X, \mu)\) is \( \#_g^\mu \) space, then for each \( x \in X \), \( \{x\} \) is either \( gs^\mu \)-closed or supra \( \alpha \)-open.

Proof. Suppose \((X, \mu)\) is \( \#_g^\mu \) space. Let \( x \in X \) and assume that \( \{x\} \) is not \( gs^\mu \)-closed, then \( X \setminus \{x\} \) is not \( gs^\mu \)-open. Then \( X \setminus \{x\} \) is the only \( gs^\mu \)-open set containing \( X \setminus \{x\} \). Thus \( X \setminus \{x\} \) is \( g_a^\mu \)-closed set. Since \( X \) is \( \#_g^\mu \) space, the set \( X \setminus \{x\} \) is supra closed. Hence \( \{x\} \) is supra open.

Definition 4.16 A space \( X \) is called a \( g_a^\mu \)-space if every \( g_a^\mu \)-closed set in it is \( g_a^\mu \)-closed.

Theorem 4.17 Every \( g_a^\mu \)-space is a \( g_a^\mu \)-space but not conversely.

Proof. Let \( X \) be a \( g_a^\mu \)-space and \( A \) be a \( g_a^\mu \)-closed set of \( X \). Since \( X \) is a \( g_a^\mu \)-space, \( A \) is \( g_a^\mu \)-closed.
Definition 5.7 The supra $\tilde{g}_\alpha$-closure of a set $A$, denoted by $\tilde{g}_\alpha^\mu$-cl(A), is the intersection of all $\tilde{g}_\alpha^\mu$-closed sets including $A$. The supra $\tilde{g}_\alpha$-interior of a set $A$, denoted by $\tilde{g}_\alpha^\mu$-int(A), is the union of all supra $\tilde{g}_\alpha$-open sets included $A$.

Remark 5.8 It is clear that $\tilde{g}_\alpha^\mu$-int(A), is a supra $\tilde{g}_\alpha$-open set and $\tilde{g}_\alpha^\mu$-cl(A), is a supra $\tilde{g}_\alpha$-closed set.

Theorem 5.9
(i) $\tilde{g}_\alpha^\mu$-cl(A) and $A = \tilde{g}_\alpha^\mu$-cl(A) if and only if $A$ is a supra $\tilde{g}_\alpha$-closed set.
(ii) $\tilde{g}_\alpha^\mu$-int(A) $\subseteq$ $A$ ; and $A$ is a supra $\tilde{g}_\alpha$-open set if and only if $A = \tilde{g}_\alpha^\mu$-int(A).
(iii) $X - \tilde{g}_\alpha^\mu$-int(A) = $\tilde{g}_\alpha^\mu$-cl(X - A)
(iv) $X - \tilde{g}_\alpha^\mu$-cl(A) = $\tilde{g}_\alpha^\mu$-int(X - A)

Proof. The proof is trivially follows from the definition.

Theorem 5.10 Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$.

If a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous, $g : (Y, \sigma) \rightarrow (Z, \eta)$ is supra $\tilde{g}_\alpha^\mu$-continuous, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is supra $\tilde{g}_\alpha^\mu$-continuous.

Proof. Since composition of two continuous functions is continuous, and since every continuous function is $\tilde{g}_\alpha$-continuous, $g \circ f$ is $\tilde{g}_\alpha^\mu$-continuous.

6. CONCLUSION

In this paper we have introduced supra $\tilde{g}_\alpha$-closed sets in supra topological spaces and established the relationship between this and other existing supra closed sets. We also defined and characterized the properties of some separation axioms using this supra $\tilde{g}_\alpha$-closed sets. Furthermore we investigated the properties of supra $\tilde{g}_\alpha$-continuous functions. We can extend this into many other research fields of General topology such as fuzzy topological spaces, ideal topological spaces and digital topological spaces.

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