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Pebbling In Watkins Snark Graph

Ms.Sreedevi S[1], Dr.M.S Anilkumar[2],
Department of Mathematics, Mahatma Gandhi College, Thiruvananthapuram, Kerala[1, 2]
Email:sreedevi.sasikumar@gmail.com[1],anilkumar250365@gmail.com[2]

Abstract- The pebbling theory is a study of a mathematical game that is played over a graph. In this game, a pebbling movement means removing two pebbles from a given vertex and adding one pebble to one of its neighbors and removing the other pebble from the game. The pebbling number of a graph G is defined a smallest positive integer required to add a pebble at any target vertex of the graph. It is denoted as $\pi(G)$. Every vertex of the graph is pebbled irrespective of the initial pattern of pebbles. Cubic graph is also called a 3- regular graph which is used in a real time scenario. In this paper, we have determined the pebbling number of Watkins Snark by constructing a Watkins Flower Snark of vertices, edges, cycles and disjoint sets which are present in the Watkins Snark. It is a connected graph in which bridgeless cubic index is equal to 4 with 75 edges and 50 vertices.

Keywords - Graphs; pebbling number; Watkins Snark graph

1. INTRODUCTION

Throughout this paper, let $G = (V, E)$ denotes a simple connected graph. $n = |V|$ and $m = |E|$ are the number of vertices and edges respectively in G and the diameter of G is denoted as d. The weights assigned to the vertices of a graph are a non-negative integer which might represent a discrete resource.

Lagarias and Saks first suggested the idea of pebbling in graphs to solve specific problems in number theory. In 1989 Chung [1] defined the pebbling number for any graph G.

1.1 *Pebbling move*

A pebbling move on a graph means taking two pebbles off from the particular vertex and placing one of the pebbles at any neighboring vertex and eliminating another pebble from the game.

1.2 *Graph pebbling*

Graph pebbling is a branch of graph theory, which is considered as a game to play on a graph with a given distribution of pebbles over the vertices of the graph.

For example, consider two players playing the mentioned game. Player A distributes the pebbles over the vertices of the graph and asks Player B to reach the target vertex by making a sequence of pebbling moves if player B reaches the target, either he wins or player A wins. In such a game, the intention of the study is to find the minimum number of pebbles required to distribute over the vertices so that the player wins

1.3 Solvability

Consider Peterson graph with pebbles on the vertices. From the (Figure-1), one can move 2 pebbles among 3 pebbles from the vertex and add 1 pebble to the neighboring vertex. Such a distribution of pebbles on vertex over a graph is solvable. However from (Figure-2), the movement of the pebble is not possible at any vertices, but every vertex contains either 0 or one pebble. Also the pebbling movement will be possible only if 2 pebbles are available at any one of the vertex.

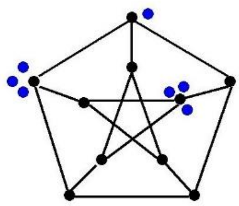

Figure-1: Solvable

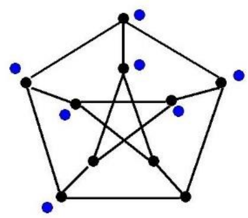

Figure-2: Unsolvable

1.4 *Pebbling number*

In a graph $G = (V, E)$, the smallest possible natural number n required to obtain a new configuration in which the target or root vertex has one or more pebbles after a series of pebbling which moves from the initial configuration. This pebbling movement is denoted by $\pi(G)$**.** It denotes the fewest number of pebbles independent from the initial configuration. Pebbling numbers of a path, cycle and wheel graphs over n vertices are given below

$$\pi(K_n) = n \qquad (Eq. 1)$$

$$\pi(P_n) = 2^{n-1} \qquad (Eq.2)$$

$$\pi(W_n) = n \qquad (Eq.3)$$

1.5 *Watkins Snark graph*

Watkins Snark Graph is a Snark in graph theory, discovered by John J. Watkins in 1989. He used 50 vertices and 75 edges to describe the snark. It denotes that J_{50} is a graph with vertex set $V(J_{50}) = V_1 \cup V_2$ $where\ V_1 = \{a_i: i = 1,2,3,4, \ldots \ldots, 25\}$, $V_2 = \{b_i: i = 1,2,3,4, \ldots, 25\} = V_5^1 \cup V_5^2 \cup V_5^3 \cup V_5^4 \cup V_5^5$

$V_5^1 = \{b_1, b_{19}, b_2, b_{20}, b_3\}$, $V_5^2 = \{b_{11}, b_4, b_{12}, b_5, b_{13}\}$
$V_5^3 = \{b_{14}, b_{22}, b_{15}, b_{23}, b_{21}\}$, $V_5^4 = \{b_{24}, b_7, b_{25}, b_8, b_6\}$

$V_5^5 = \{b_9, b_{17}, b_{10}, b_{18}, b_{16}\}$

And edge set $E = E_1 \cup E_2$

$$where\ E_1 = \{\, e_i^1 = a_i\, a_{i+1} : i = 1,2,3, \ldots., 24\}$$

$$E_2 = \{e_i^1 = a_i b_i : i = 1,2,3, .., 25\}$$

$$\cup\ E_5^1 \cup E_5^2 \cup E_5^3 \cup E_5^4 \cup E_5^5$$

where $E_5^1 = \{b_1 b_{19}, b_{19} b_2, b_2 b_{20}, b_{20} b_3, b_3 b_1\}$,

$$E_5^2 = \{b_{11} b_4, b_4 b_{12}, b_{12} b_5, b_5 b_{13}, b_{13} b_{11}\},$$

$$E_5^3 = \{b_{21} b_{14}, b_{14} b_{22}, b_{22} b_{15}, b_{15} b_{23}, b_{23} b_{21}\}$$

$$E_5^4 = \{b_6 b_{24}, b_{24} b_7, b_7 b_{25}, b_{25} b_8, b_8 b_6\},$$

where V_5^1, V_5^2, V_5^3, V_5^4, V_5^5 and E_5^1, E_5^2, E_5^3, E_5^4, E_5^5 are disjoint sets of vertices and edges of cycles C_5^1, C_5^2, C_5^3, C_5^4, C_5^5 respectively. In Fiure-3, Watkins Snark set of vertices V_1 forms a cycle of C_{25} numbers.

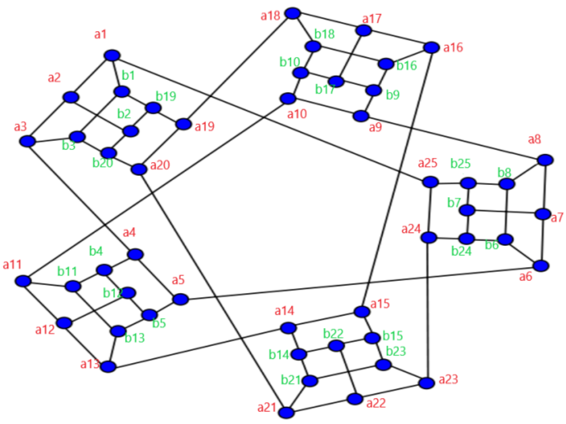

Figure-3: Watkins Vertex set

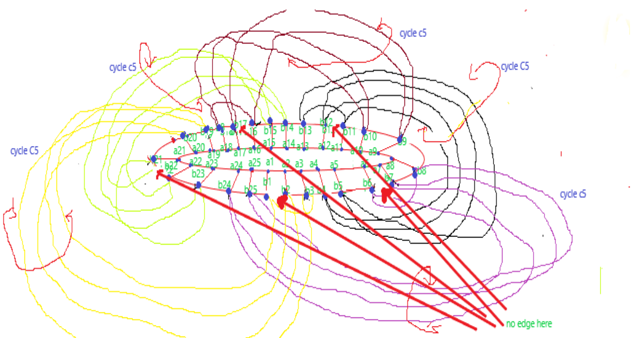

Figure-4: Flower Watkins Snark (Edge set)

Pebbling on graphs is a two-player game on a connected graph, first introduced by Lagarias and Saks. Later Chung [1] gave the new introduction for the two-player game. In [2], Pachter et al. proved that every graph of diameter two on N vertices has a pebbling number either N or N +1.. In [3], A. Lourdusamy et al. introduced the generalized pebbling number of a graph. David Moews [4] proved that the pebbling number of the product of two graphs $f(G \times H) \leq f(G)f(H)$.Ye et al. [5] determined the pebbling numbers of squares of even cycles. Whereas Asplund et al. [6] showed $\pi(G \times H) \leq (\pi(G) + |G|)\pi(H)$ and provided similar results for other graph products and graph operations. Melody Chan et al. [7] upper bounded by considering the configuration of pebbles distributed on the vertices of a connected graph of order n. If G is a connected graph with n vertices and $\delta(G) = k$, then optimal pebbling $\pi^*(G) \leq \frac{4n}{k+1}$. For Watkins Snark $J_{50}, k = 3\ and\ n = 50$,

$$\pi^*(J_{50},) \leq \frac{4\times 50}{3+1} = 50 \text{ thus } \pi^*(J_{50},) = 50.$$

$$max\{n, 2^d\} \leq \pi(G) \leq (n-1)(2^d - 1) + 1 \qquad (Eq.4)$$

(2), Melody Chan et al. [7] proposed the new upper bound as,

$$\pi(G) \leq (n-d)(2^d - 1) + 1 \qquad (Eq.5)$$

In (3),Melody Chan et al. [7] improved the upper bound of the theorem (2)

$$f(G) \leq \left(n + \left\lceil \frac{n-1}{d} \right\rceil - 1\right)(2^{d-1}) - n + 2 \qquad (Eq.6)$$

In (3),Melody Chan et al. [7]proved that G has an efficient dominating set of size γ.

2. LOWER AND UPPER BOUNDS OF PEBBLING NUMBER OF WATKINS SNARK

Watkins graph has $d = diam(J_{50}) = 7$ and size n = 50. Thus

$$max\{50, 2^7\} \leq \pi(J_{50}) \leq (50-1)(2^7-1)+1$$

$$max\{50,128\ \} \leq \pi(J_{50}) \leq (49)(127)+1$$

$$128 \leq \pi(J_{50}) \leq 6224 \qquad \text{(Eq.7)}$$

From $Eq.4$, improved bounds become

$$\pi(G) \leq (50-7)(2^7-1)+1 = 5462$$

$$128 \leq \pi(J_{50}) \leq 5462 \qquad (Eq.8)$$

From $Eq.5$, new bounds become

$$\pi(J_{50}) \leq \left(50 + \left\lfloor \frac{50-1}{7} \right\rfloor - 1\right)(2^{7-1}) - 50 + 2$$

$$\pi(J_{50}) \leq 3536$$

$$128 \leq \pi(J_{50}) \leq 3536 \qquad (Eq.9)$$

Theorem.2.1. In (5), Bukh Boris [8] proved that pebbling number of a graph with n vertices and diameter d satisfies

$$\pi(n,d) \leq (2^{\lceil \frac{d}{2} \rceil} - 1)n + O(\sqrt{n}) \qquad (Eq.10)$$

Theorem.2.2. In (7), Postle [9]showed that if diameter d of a connected graph with n vertices is odd, then

$$\pi(G) \leq f\left(\left\lceil \frac{d}{2} \right\rceil\right) n + O(1), \text{where f(k)} = \frac{(2^k-1)}{k} (Eq.11)$$

For Watkins Snark J_{50},

$$\pi(J_{50}) \leq f\left(\left\lceil \frac{7}{2} \right\rceil\right) 50 + O(1) = 50\, f(4) + O(1),$$

$$\text{f(4)} = \frac{(2^4-1)}{4} = \frac{15}{4}.$$

$$\pi(J_{50}) \leq 50\left(\frac{15}{4}\right) + O(1) = 187.5 + O(1)$$

$$128 \leq \pi(J_{50}) \leq 187.5 + C \qquad (Eq.12)$$

where C is a constant.

3. PEBBLING NUMBER OF WATKINS SNARK

3.1 *Pebbling number of Watkins Snark J_{50}*.

A Watkins Snark J_{50} is a graph whose vertex set is partitioned into two disjoints subsets $V_1 and\ V_2$ where $V_1 = \{a_{1,}, a_2, a_3, \dots \dots a_{25}\}$ and

$$V_2 = \{b_{1,}, b_2, b_3, \dots \dots b_{25}\}.$$

Set V_2 can be again portioned as

$V_2^1 \cup V_2^2 \cup V_2^3 \cup V_2^4 \cup V_2^5$ where

$$V_2^1 = \{b_1, b_{25}, b_{24}, b_{23}, b_{22}\}$$

$$V_2^2 = \{b_2, b_3, b_4, b_5, b_6\}$$

$$V_2^3 = \{b_7, b_8, b_9, b_{10}, b_{11}\},$$

$$V_2^4 = \{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}\}$$

$$V_2^5 = \{b_{17}, b_{18}, b_{19}, b_{20}, b_{21}\}.$$

Every vertex $a_i \epsilon V_1$ is adjacent to vertex $b_i \epsilon V_2$ for $all\ 1 \leq i \leq 25$.The vertex sets of the cycle C_5 are given below:

$V_{C_5}^1 = \{b_1, b_{19}, b_2, b_{20}, b_3\}$, $V_{C_5}^2 = \{b_{11}, b_4, b_{12}, b_5, b_{13}\}$, $V_{C_5}^3 = \{b_{14}, b_{22}, b_{15}, b_{23}, b_{21}\}$,

$V_{C_5}^4 = \{b_{24}, b_7, b_{25}, b_8, b_6\}$, $V_{C_5}^5 = \{b_9, b_{17}, b_{10}, b_{18}, b_{16}\}$ are subsets of V_2 .

Denoting the number of pebbles distributed over each vertex of V_j for all $j = 1,\ 2$ by p_j, equation A.12 is given as,

$$p_2 = p_2^1 + p_2^2 + p_2^3 + p_2^4 + p_2^5 \qquad (Eq.13)$$

where p_2^jdenotes the number of pebbles distributed over each vertex of $V_2^j for\ all\ j = 1, 2, 3, \dots, 5$.

Let the number of pebbles initially placed on a particular vertex x_j is denoted by p(x_j) for all $1 \leq j \leq 50$. Let the total number of pebbles on the set V_j for all $j = 1,\ 2$ is denoted by $p(j)$. So

$$p(2) = p^1(2) + p^2(2) + p^3(2) + p^4(2) + p^5(2) \qquad (Eq.14)$$

where $p^j(2)$ denotes the total number of pebbles on the set V_2^j for all $j = 1, 2, 3, \dots, 5$. There are two possible cases of sets where the target vertex may belong $V_1 and\ V_2$.

Case 1:

In this case, the target vertex $a_1 \in V_1$. If $p(a_2) = 2$ or $p(a_{25}) = 2$ determines the pebbling movement either from a_2 or a_{25}, then only 2 pebbles are necessary to reach the target that is trivial. So assume that

$$p(a_2) < 2 ,\ p(a_{25}) < 2 \qquad (Eq. 15)$$

***Case 1.1*: $p_1 > 1$:**

In this case, assume that the vertex of the set V_1would distribute at least two pebbles in the vertex. Then for some$i$,such that $p(a_i) \geq 2\ for\ 3 \leq i \leq 24$ which would start the first step in pebbling. Then the pebbling sequence would reach the target vertex which is given as$\{a_i, a_{i-1}, \dots, a_2, a_1\}$. On placing two pebbles on each of $25 - 3 = 22$ vertices of V_1 and in particular at least one pebble on a_2 and a_{25},the sum of the total number of pebbles required to reach the target vertex a_1 is

$$p^1 \geq 2 \times 22 + 2 = 46 \qquad (Eq. 16)$$

The target can also be reached by placing 2^d pebbles on a single vertex, where d is the diameter to the target vertex.Since set V_1 forms an odd cycle C_{25}, the required number of pebbles $V_1$to pebble the target vertex a_1 is given by

$$\pi(C_{25}) = 2\left\lfloor\frac{2^{d+1}}{3}\right\rfloor + 1 = 2\left\lfloor\frac{2^{12+1}}{3}\right\rfloor + 1 = 5461 \qquad (Eq. 17)$$

***Case 1.2*: $p_1 \leq 1$:**

In this case, the set V_1 has insufficient number of pebbles to start a pebbling step. In order to reach the target a_1, we need to extract the pebbles from the set V_2. The following sub cases are possible ways to extract the pebbles from V_2. In these sub cases, assume that $p(b_1) = 0$ which avoids the trivial pebbling numbers.

***Case 1.2.1*: $p_2 \geq 2$:**

In this case, suppose $p_2 \geq 2 \Rightarrow p_2^j \geq 2$ for all j =1,2,3,4,5**,** then each vertex a_i is adjacent to vertex b_i for all $i \in \{1,2,\dots 25\}$. Consider $p(b_1) = 0$, since $p_2 \geq 2$, one pebble can be moved to V_1 so that $p_1 \geq 2$. After these pebbling moves, the target vertex is reached using the case 1.1 When $p_1 = 0$, there will be no vertex in V_1 in order to make the pebbling movement. Hence to start the pebbling movement i should be in $2 \leq i \leq 25$ such that $p(b_i) \geq 4$ allows two pebbles moved to vertex $a_i \epsilon V_1$ and the sequence of pebbling moves would be $\{b_i, a_i, a_{i-1}, a_{i-2}, \dots a_2, a_1\}$ or

$\{b_i, a_i, a_{i+1}, a_{i+2}, \dots a_{25}, a_1\}$ for $2 \leq i \leq 25$. Thus there are at most one pebble on 24 vertices of V_1 and minimum two pebbles on 24 vertices of V_2, so the total number of pebbles required to reach the target vertex is

$$p^1 + p^2 \geq 24 + 2 = 72 \qquad (Eq. 18)$$

***Case 1.2.2*: $p_2^1 \leq 1$**

Since $p_2^1 \leq 1$, set V_2^1 has insufficient number of pebbles to move one pebble to some vertex of V_1.

To make a pebbling move a set of V_2^k pebbles must be extracted where $k\ \boldsymbol{\epsilon}\ \{\mathbf{2, 3, 4, 5}\}$ which is discussed in following sub cases.

***Case 1.2.2.1*: $p_2^k \geq 2$ for some $k \epsilon \{2, 3, 4, 5\}$:**

As every $b_i \epsilon V_2$ belongs to cycle C_5, for example, $b_1 \epsilon C_5 = \{b_1, b_{19}, b_2, b_{20}, b_3\}$.Through this cycle the pebbles can be extracted from V_2^l for some $l \epsilon \{2, 3, 4, 5\}$ by making the pebbling sequence of length at most 5, i.e. a path of length 5

Assume $p(x_i) \geq 2\ for\ each\ x_i\ \epsilon\ \{b_1, b_{19}, b_2, b_{20}, b_3\}$

We know pebbling number of a path P_5 is given by

$$\pi(P_5) = 2^{5-1} = 2^4 = 16 \qquad (Eq. 19)$$

Which denotes the required number of pebbles to pebble the target vertex x_1 is greater than the minimum number of pebbles. Therefore it is found that there are at most one pebble in 25 vertices of V_1 and if 16 pebbles are placed at the path, 1 pebble is distributed at each vertex of V_1 except the cycle. Hence the minimum number of pebbles required are $p_1 + 20 + 16 = 25 + 36 = 61$. If

$$p(x_i) \leq 1\ for\ each\ x_i\ \epsilon\ \{b_1, b_{19}, b_2, b_{20}, b_3\}.$$

then $p(x_i) = 1$. In order to move a pebble to a_1 the pebbles have to be extracted from other cycles, $C_5 \neq \{b_1, b_{19}, b_2, b_{20}, b_3\}$. From the extracted pebbles from other cycles, a pebbling sequence of 8 vertices is needed i.e., a path of length 8 and the known pebbling number of a path length n is $\pi(P_n) = 2^{8-1}$=$2^7 = 128$.

In this case, 1 pebble at 25 vertices of V_1, 1 pebble at each vertex of the cycle and 128 pebbles at the path of 8 vertices and 1 pebble at each of remaining 13 vertices of V_2. Hence, in this case, the minimum number of pebbles required is

$$p_1 + 1 \times (13) + 128 = 25 + 1 \times 13 + 128 = 166 \qquad (Eq. 20)$$

Let $p(x_i) = 0$, in this case for each $x_i \in \{b_1, b_{19}, b_2, b_{20}, b_3\}$, there will be no vertex in order to make a pebbling movement. The pebbling sequence must be through the vertices of V_1 which have 8 vertices. For example to reach target vertex a_1 and all the elements at the cycle having 0 pebble a pebbling sequence $\{b_{19}, a_{19}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22,} a_{23}, a_{24}\}$ is made.

Case 2:

In the second possibility of the target, vertex would be a vertex set which is identified as V_2. So let $b_1 \epsilon V_2$ be the target vertex. Each vertex $b_i \epsilon V_2$ is adjacent to 5 and vertices except $b_1, b_2, b_6, b_7, b_{11}, b_{12}, b_{16}, b_{17}, b_{21}, b_{22}$ are adjacent to 4 vertices. Our target vertex b_1 is adjacent to 4 vertices. So assumption can be made to avoid the trivial pebbling step as follows,

$$p(a_1) = 1, p(b_3) = 1, p(b_{19}) = 1, \; p(b_{25}) = 1 (Eq. 21)$$

***Case 2.1*: $p_2 \geq 2$**

In this case, pebbles are to be transmitted from any arbitrary vertex $b_i \epsilon V_2$ to reach the target vertex $b_1 \epsilon V_2$. Each vertex $b_i \epsilon C_5$ is adjacent to vertex a_i for $i = 1, 2, 3, \ldots\ldots, 25$. In order to start the pebbling move, it is necessary to extract the pebbles from V_1. So the minimum number of pebbles required can be denoted as the equation Eq.22.

$$p_1 + p_2 = 2 \times 24 + 1 \times 22 = 70 \qquad (Eq. 22)$$

Case 2.2:$p_2 \leq 1$

When $\boldsymbol{p_2 = 1}$, pebbles will be extracted from V_1. Assume $\boldsymbol{p_1 = 2}$, then one pebble can be moved to the vertices of V_2 and the target vertex is pebbled. So the required minimum number of pebbles is given as,

$$p_1 + p_2 = 22 + 2 \times 24 + 4 = 74 \qquad (Eq. 23)$$

Let $\mathrm{p}_2 = 0$. This allows two pebbles moved to b_1. In this case, pebbles can be moved through a cycle C_5 from V_1. Pebbling number of C_5 is

$$\pi(C_5) = 2\left\lfloor \frac{2^{2+1}}{3} \right\rfloor + 1 = 5 \qquad (Eq. 24)$$

So the required minimum number of pebbles is

$$p_2 + p_1 = 2 \times 24 + 5 = 52 \qquad (Eq. 25)$$

***Case 2.2.1*:$p_2 \leq 1$**

Here, extraction of pebbles takes place from V_1**.** In this case, V_2 have an insufficient number of pebbles. So we need to extract pebbles from the set V_1. Assume $p_2^1 \leq 1$. In this case, V_2^1 has the insufficient number of pebbles to make pebbling move to target vertex b_1. So we can extract from $\boldsymbol{V_1}$ through cycle C_5. Hence we need to make a pebbling sequence of 8 vertices, i.e. a path of length 8 and minimum number of pebbles required in a path graph length n is

$$\pi(P_n) = 2^{8-1} = 2^7 = 128 \qquad (Eq. 26)$$

In this case, 1 pebble is at 25 vertices of V_2, 1 pebble at each vertex of the cycle and 128 pebbles at the path of 8 vertices and 1 pebble at each of remaining 13 vertices of V_2. Hence the minimum number of pebbles required as the target pebble is

$$p_1 + 1 \times (13) + 128 = 25 + 1 \times 13 + 128 = 166 \quad (Eq. 27)$$

On comparing the case (2.1) and (2.2), the set of possible minimum pebbling numbers is obtained as $\{46, 72, 61, 166\}$ in case (2.1). In case (2.2) it is obtained as $\{70, 52, 166\}$.From the equation (*Eq. A.29*), as all the possibilities of pebbling the target vertex in the sets V_1, V_2 are discussed, the equation becomes

$$128 \leq \pi(J_{50}) \leq 187.5 + C \qquad (Eq. 28)$$

On considering the least possibility of the pebbling number from the two cases, it is concluded that the least possibility of $\{46, 72, 61, 166\}$ and $\{70, 52, 166\}$ is 166. Thus

$$\pi(J_{50}) = 166 \qquad (Eq. 29)$$

4. CONCLUSIONS

In this study, the pebbling number of Watkins Snark is calculated by constructing a Watkins Flower Snark by edges, vertices, cycles and disjoint sets present in the graph. By applying the methodology, the pebbling number of graphs is determined as Double-star snark, Szekeres snark, Loupekine snark, Fullerene graphs. The applications of pebbling are in positional games such as “Cops-and-Robbers" and "Chip-Firing." The pebbling methodology of these games can be used in structural graph theory and theoretical computer science. Theory of pebbling can be applied in a toll or as a loss of information, fuel or electrical charge. The concept of graph pebbling can be generalized as q-pebbling and as the rate of loss. We can choose any constant rate α of loss instead of integer values in the initial distribution of pebbles. In a pebbling step one removes weight w from one vertex and places weight αw at an adjacent vertex, for any constant $0 < \alpha < 1$. The objective of the study is still to place weight 1 at any prescribed root r so that there is enough money, fuel, information, or energy at that location in the network. This generalized α-pebbling is useful to make a chip-firing model. For any graph G, an auxiliary graph H can be obtained, so that chip-firing results on H can be brought on pebbling number of G.

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