Dynamic Analysis of a Fractional Order Three Species Interaction Model with Holling Type II Response

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Abstract- This paper considers a three species predator-prey model with Holling type II response in fractional order . The model is investigated with the assumption that apex predator consumes both the logistically growing prey and the mesopredators. Stability properties and the existence of periodic solutions are examined for the equilibrium points and the results are strengthened with the numerical simulations. Time evolution of the Lyapunov Exponents of the fractional order system is tabulated and dynamics of the Lyapunov Exponent is discussed with the 2-D plot. Also, the effect of the fractional order on the dynamics of the system is analyzed. Time plots, phase plane graphs and bifurcations diagrams are presented.

Index Terms- Fractional differential equation; Discretization; Population Dynamics; Fixed Points; Stability; Bifurcation;

1. INTRODUCTION

Fractional calculus has its origin during the 17th century, but it is 20th century when most of the theory applicable to study Fractional calculus has been developed. Despite the long history, motivation for the researchers to work on fractional calculus has been increasing over a decade. Especially, lots of works are carried out on fractional order systems and its applications in various fields and recent developments on the theory on existence and uniqueness, stability conditions of solutions of non-linear FDEs. [[3],[5]]

In discussion of the stability properties of fractional order systems the following lemmas are used.

Lemma 1.1: [[4],[7]] The following linear commensurate fractional - order autonomous system
$$D^\alpha x = Ax, x(0) = x_0$$
is asymptotically stable if and only if $|arg\lambda| > \alpha \frac{\pi}{2}$ is satisfied for all eigenvalues of matrix A. Also, this system is stable if and only if $|arg\lambda| > \alpha \frac{\pi}{2}$ is satisfied for all eigenvalues (λ) of matrix A and those critical eigenvalues which satisfy $|arg\lambda| = \alpha \frac{\pi}{2}$ have geometric multiplicity one, where $0 < \alpha < 1$, $x \in R^n$ and $A \in R^{n \times n}$

Lemma 1.2: [[4],[7]] The fixed points of the following linear commensurate fractional - order autonomous system

is asymptotically stable if all eigenvalues (λ) of the Jacobian matrix $\frac{\partial f}{\partial x}$ evaluate at the fixed points satisfy $|arg\lambda| > \alpha \frac{\pi}{2}$ where $0 < \alpha < 1$, $x \in R^n$ and $f: R^n \to R^n$ are continuous linear vector functions and the fixed points of this nonlinear commensurate fractional order system are calculated by solving equation $f(x) = 0$.

2. BACKGROUND OF THE MATHEMATICAL MODEL

Major In this paper, we consider an ecosystem of three species interaction with Holling type II functional response. Prey, Meso predators and Apex predators are modelled based on the assumption that Apex predator survives on predation of both logistically growing prey and Meso predators. Meso predators depends only on prey for its survival.

Main purpose of modelling a ecological system with a functional response is to represent the consumers rate of intake as a function of density of availability of food in that ecosystem. Thus, functional response of Holling type II is introduced to the system to decelerate the rate of intake by the predators (or) the predators food processing capacity is limited.

System of Fractional order differential equations representing the three species interaction is given by

$$D^{\gamma_1}x(t) = x(1-x(t)) - \frac{ax(t)y(t)}{1+x(t)} - \frac{cx(t)z(t)}{1+x(t)};$$

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$$D^{\gamma_2}y(t) = \frac{ax(t)y(t)}{1+x(t)} - \frac{by(t)z(t)}{1+y(t)} - dy(t);$$

(1) $\quad D^{\gamma_3}z(t) = \frac{cx(t)z(t)}{1+x(t)} + \frac{by(t)z(t)}{1+y(t)} - ez(t);$

where $x(t), y(t)$ and $z(t)$ represents the prey density, Meso predator density and Apex predator density respectively with γ_1, γ_2 and γ_3 being their fractional orders. In Biological populations, all the parameters of the system (1) are positive.

Symbols	Meaning
r	Growth rate of the prey
b	Interaction rate of meso predator and Apex predator
d	Natural death rate of the meso predator
a	Interaction rate of Prey and meso predator
c	Interaction rate of prey and Apex predator
e	Natural death rate of the Apex predator

3. FIXED POINT AND STABILITY ANALYSIS OF THE SYSTEM (1)

Displayed The non-negative Axial fixed points of the Fractional order system (1) are

- $E_0 = (0,0,0)$ (trivial)

- $E_1 = (1,0,0)$ (Predator free)

- $E_2 = (\frac{e}{c-e}, 0, \frac{r(c-2e)}{(c-e)^2})$ (Non existence of Meso predators)

- $E_3 = (\frac{d}{a-d}, \frac{r(a-2d)}{(a-d)^2}, 0)$ (Non existence of Apex Predator)

The Jacobian matrix of the Fractional order system (1) for equilibrium $E = (x, y, z)$ is

J

$$= \begin{bmatrix} r(1-x) - rx - \frac{ay}{1+x} + \frac{axy}{(1+x)^2} - \frac{cz}{1+x} + \frac{cxz}{(1+x)^2} & -\frac{ax}{1+x} & -\frac{cx}{1+x} \\ \frac{ay}{1+x} - \frac{axy}{(1+x)^2} & \frac{ax}{1+x} - \frac{bz}{1+y} + \frac{byz}{(1+y)^2} - d & -\frac{by}{1+y} \\ \frac{cz}{1+x} - \frac{cxz}{(1+x)^2} & \frac{bz}{1+y} - \frac{byz}{(1+y)^2} & \frac{by}{1+y} + \frac{cx}{1+x} - e \end{bmatrix}$$

3.1 Jacobian Matrix for the Fixed points

Trivial fixed point (E_0)

$$J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -e \end{bmatrix}$$

The Eigenvalues of the matrix $J(E_0)$ are $\lambda_1 = r, \lambda_2 = -e$ and $\lambda_3 = -d$

Predator free Equilibrium (E_1)

$$J(E_1) = \begin{bmatrix} -r & -\frac{a}{2} & -\frac{c}{2} \\ 0 & \frac{a}{2} - d & 0 \\ 0 & 0 & \frac{c}{2} - e \end{bmatrix}$$

$\lambda_1 = -r, \lambda_2 = \frac{c}{2} - e$ and $\lambda_3 = \frac{a}{2} - d$ are the eigenvalues of the matrix $J(E_1)$.

Meso Predator free Equilibrium (E_2)

$J(E_2) =$
$$\begin{bmatrix} -2\frac{e^2 r}{c(c-e)} & -\frac{ae}{c} & -e \\ 0 & \frac{ac^2 e - 2\,ace^2 + ae^3 - brc^2 + 2\,brce - c^3 d + 2\,c^2 de - cde^2}{(c-e)^2 c} & 0 \\ \frac{r(c-2e)}{c} & \frac{br(c-2e)}{(c-e)^2} & 0 \end{bmatrix}$$

The corresponding eigenvalues are

$$\lambda_{1,2} = \frac{-re^2 \pm \sqrt{-c^4 e\,r + 4\,c^3 e^2 r - 5\,c^2 e^3 r + 2\,ce^4 r + e^4 r^2}}{(c-e)c} \text{ and}$$

$$\lambda_3 = \frac{ac^2 e - 2\,ace^2 + ae^3 - bc^2 r + 2\,bce\,r - c^3 d + 2\,c^2 de - dce^2}{c(c^2 - 2\,ce + e^2)}.$$

Apex Predator free Equilibrium (E_3)

$J(E_3)$
$$= \begin{bmatrix} -2\frac{d^2 r}{a(a-d)} & -d & -\frac{cd}{a} \\ \frac{r(a-2d)}{a} & 0 & -\frac{br(a-2d)}{a^2 - 2\,ad + ar + d^2 - 2\,rd} \\ 0 & 0 & F \end{bmatrix}$$

where

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$$F = -\frac{a^3e - bra^2 - a^2cd - 2\,a^2de + a^2e\,r + 2\,brad + 2\,acd^2 - acdr + ad^2e - 2\,ade\,r - cd^3 + 2\,cd^2r}{a(a^2 - 2\,ad + ar + d^2 - 2\,rd)}$$

$$\lambda_{1,2} = \frac{-d^2r \pm \sqrt{-a^4dr + 4\,a^3d^2r - 5\,a^2d^3r + 2\,ad^4r + d^4r^2}}{(a-d)a} \text{ and}$$

$$\lambda_3 = -\frac{a^3e - a^2br - a^2cd - 2\,a^2de + a^2e\,r + 2\,abdr + 2\,acd^2 - acdr + ad^2e - 2\,ade\,r - cd^3 + 2\,cd^2r}{a(a^2 - 2\,ad + ar + d^2 - 2\,rd)}$$

are the eigenvalues of the matrix $J(E_3)$.

4. NUMERICAL EXAMPLES

For the parameter values
$r = 0.35, a = 0.27, \quad b = 0.01, c = 0.5, d = 0.2,$
$\qquad e = 0.02$
and the initial condition is (0.05,0.02,0.5), the

The condition $|arg\lambda| > \alpha\frac{\pi}{2}$ is satisfied by all the eigen values,thus the system is asymptotically stable.The stability of the system for each species density is explained in Figure (1) and the 2-D and 3-D phase portrait is also given.
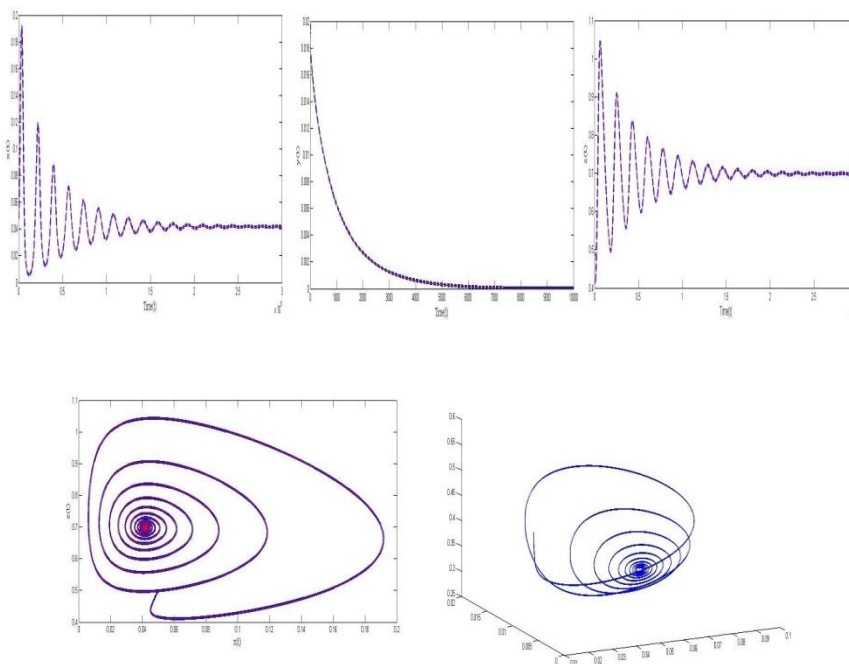

Fig. 1. Time plots and Phase portrait for the system (1).

commensurate fractional order system (1) with order
$$\gamma_1 = \gamma_2 = \gamma_3 = 0.97$$

has eigen values
$$\lambda_1 = -0.0005833333332 \pm 0.08024749045\ i$$
and
$$\lambda_3 = -0.1961878472.$$

Figure (2) explains the Limit cycle of the the commensurate fractional order system (1) with order
$$\gamma_1 = \gamma_2 = \gamma_3 = 0.9$$
and initial condition is (0.45,0.35,0.15) with parameter values
$$r = 2.99, a = 0.82, b = 0.64, c = 0.2, d = 0.18,$$
$$e = 0.4$$
.

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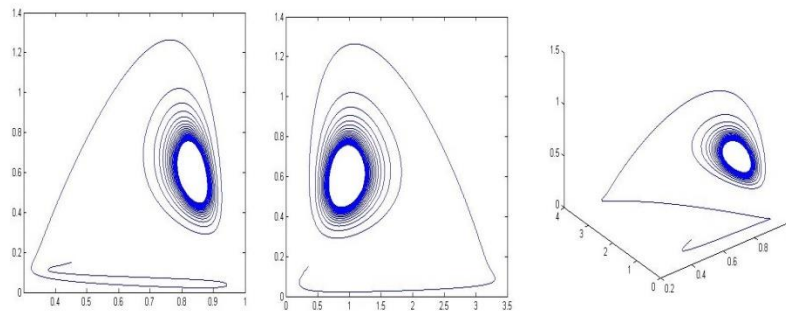*Available online at www.ijrat.org*

Fig. 2. Limit cycle for the system (1).

5. DISCRETIZATION OF THE SYSTEM (1)

Applying piecewise constant arguments methods [1] to the commensurate fractional order system (1) of order γ, the discretized fractional order system is of the form

$$
\begin{aligned}
x(t+1) &= x(t) + L\left[r(1-x(t))x(t) - \frac{ax(t)y(t)}{1+x(t)} - \frac{cx(t)z(t)}{1+x(t)}\right]\\
y(t+1) &= y(t) + L\left[\frac{ax(t)y(t)}{1+x(t)} - \frac{by(t)z(t)}{1+y(t)} - dy(t)\right]\\
z(t+1) &= z(t) + L\left[\frac{cx(t)z(t)}{1+x(t)} + \frac{by(t)z(t)}{1+y(t)} - ez(t)\right]
\end{aligned}
$$
(2)

where γ is the fractional order, $L = \frac{h^{\gamma}}{\Gamma(1+\gamma)}$ and h is the step size of discretization

6. BIFURCATION OF THE SYSTEM (2)

The Bifurcation is the study of the changes that occur in the system when a value of the parameter is varied. Bifurcation diagrams helps in understanding the chaotic nature of the system.[2] In this paper, the bifurcation of the system (3) is obtained for the growth rate of the prey (r) and the order (γ) of the system (2).

6.1. Growth rate as Bifurcation parameter

For the fixed values of parameters $h = 1.13, \gamma = 0.85, a = 0.82, b = 0.64, c = 0.2, d = 0.18, e = 0.4$, varying the growth rate (r) in the range 0 to 3, initial condition is $(0.45, 0.35, 0.15)$, the bifurcation diagrams in the three dimensions are obtained for the bifurcation parameter as shown in the Figure(3).
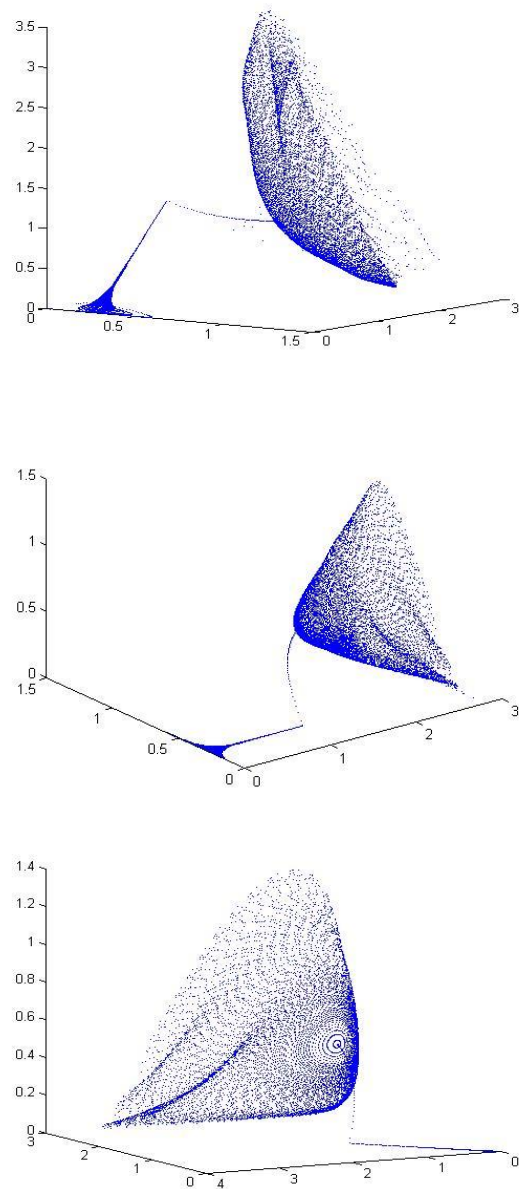

Figure 3: Bifurcation Parameter (r)

6.2.Fractional order as Bifurcation Parameter

Considering the range of the Bifurcation parameter range in between 0 to 2, the diagram in Figure (4) is simulated for the fixed parametric values
$$h = 1.13, r = 1.99, a = 0.82, b = 0.64, c = 0.2,$$
$$d = 0.28, e = 0.4$$
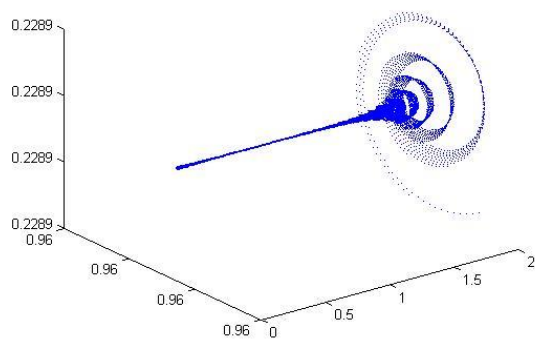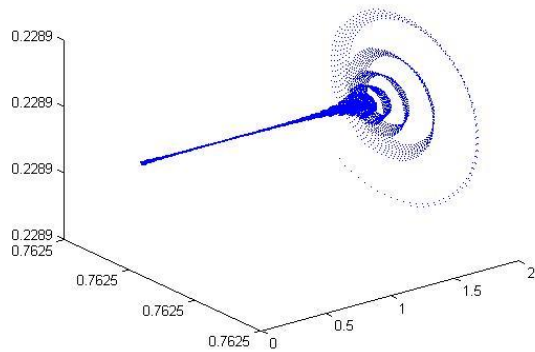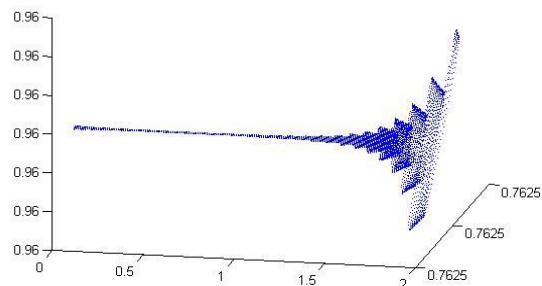with initial condition (0.45,0.35,0.15).

7. LYAPUNOV EXPONENTS

Any two orbits starting from nearby initial points may either converge or diverge.[6] The measure of their average rate of convergence or divergence is the Lyapunov exponents. In this paper, we obtain the Lyapunov exponents as a function of time and the values used for plotting is provided in table [1].

Figure (6) describes the Lyapunov exponents of the fractional order system (1) with commensurate fractional order 0.92 with the parameters $r = 0.75, a = 0.8, b = 51, c = 0.64, d = 0.1, e = 0.4$ and initial conditions (0.008,0.009,0.007).

Table 1. Time Evolution of LE.

Time	LE1	LE2	LE3
10	0.1729	-0.0940	-0.2806
20	0.1294	-0.1132	-0.3444
30	0.0652	-0.1007	-0.2830
40	0.0532	-0.0843	-0.2326
50	0.0375	-0.0708	-0.2192
60	0.0352	-0.0770	-0.1952
70	0.0282	-0.0620	-0.1889
80	0.0234	-0.0664	-0.1763
90	0.0202	-0.0561	-0.1712
100	0.0133	-0.0548	-0.1647
110	0.0097	-0.0479	-0.1600
120	0.0033	-0.0434	-0.1566
130	0.0008	-0.0403	-0.1527
140	0.0014	-0.0408	-0.1505
150	0.0023	-0.0423	-0.1475
160	0.0019	-0.0412	-0.1457
170	0.0007	-0.0408	-0.1436
180	-0.0008	-0.0387	-0.1420
190	-0.0032	-0.0367	-0.1405
200	-0.0057	-0.0340	-0.1391

Figure 4: Bifurcation Parameter (γ)
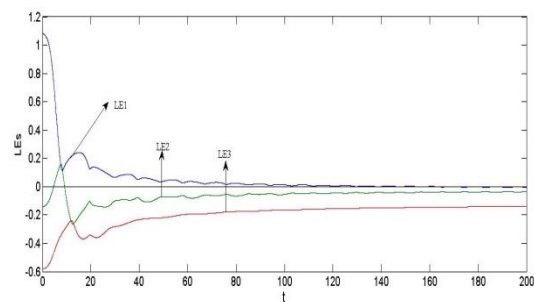
Figure 5: Lyapunov Exponent of the system (1)

8. CONCLUSION

The paper investigated the stability properties of the non-linear fractional order dynamical system with holling type II response. Time plots and phase portraits are provided to explain the dynamics of the system. The bifurcation of the system in its equivalent discretized form is analyzed with 2-D and 3-D bifurcation diagrams. Growth parameter and fractional order considered as bifurcation parameter are used to study the chaotic properties. The Lyapunov exponents are calculated and plotted against time. The fractional order systems clearly exhibits rich dyanmics of a non linear dynamical system.

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